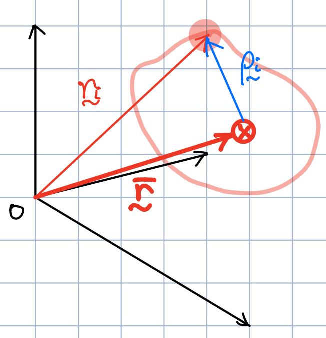
3. Impulse & Momentum

(a) Linear Momentum



1. one particle : $G_i = m_i v_i = m_i \dot{r}_i$

2. System of particles : $G = \Sigma G_i = \Sigma m_i \dot{r}_i$ ← $r_i = \bar{r} + \rho_i$

$= \Sigma m_i (\dot{\bar{r}} + \dot{\rho}_i)$

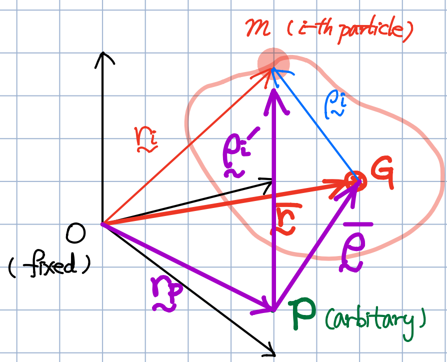
$= \Sigma m_i \dot{\bar{r}} + \cancel{\frac{d}{dt}(\Sigma m_i \rho_i)}$ ← $\Sigma m_i r_i = (\Sigma m_i)\bar{r}$

$\Sigma m_i (r_i - \bar{r})$

$= \Sigma m_i \rho_i = 0$

$\boxed{G = M \dot{\bar{r}}}$

(b) Angular Momentum



★점에서 관측한 △에 대한 각운동량

★	△	Answer
O	O	$H_O = r_i \times m_i \dot{r}_i$
O	G	$H_G = \rho_i \times m_i \dot{r}_i$
G	G	$(H_G)_{relG} = \rho_i \times m_i \dot{\rho}_i$
P	G	$(H_G)_{relP} = \rho_i \times m_i \dot{\rho}_i'$

1. (O점에서 관측한) O점에 대한 각운동량

$H_O = \sum_i (r_i \times m_i \dot{r}_i)$

$\dot{H}_O = \Sigma (\cancel{\dot{r}_i \times m_i \dot{r}_i}) + \Sigma r_i \times m_i \ddot{r}_i$

$\dot{r}_i // \dot{r}_i$ $\quad$ $= F_i$

$= \Sigma r_i \times F_i = \Sigma M_i$

고정점 O 에 대한 Resulting Moment는

System O점에 대한 각운동량 변화율과 같다.

$\dot{H}_G = \Sigma \rho_i \times m_i \ddot{r}_i$

$= \Sigma \rho_i \times f_i = \Sigma M_G$

2. CM에 대한 각운동량

$(H_G)_{relG} = \Sigma \rho_i \times m_i \dot{\rho}_i = \Sigma \rho_i \times m_i (\dot{r}_i - \dot{\bar{r}})$

$= \Sigma \rho_i \times m_i \dot{r}_i - (\Sigma \rho_i \times m_i \dot{\bar{r}})$ ← $\alpha u \times \beta v = \alpha\beta (u \times v)$

$= \beta u \times \alpha v$

$= \Sigma \rho_i \times m_i \dot{r}_i - (\cancel{\Sigma m_i \rho_i}) \times \dot{\bar{r}}$

$= \Sigma \rho_i \times m_i \dot{r}_i$

$= H_G$ (O 점에서 관측한 G에 대한 각운동량)

$(H_G)_{relP} = \Sigma \rho_i \times m_i \dot{\rho}_i' = \Sigma \rho_i \times m_i (\dot{r}_i - \dot{r}_p)$

$= \Sigma \rho_i \times m_i \dot{r}_i - \Sigma \rho_i \times m_i \dot{r}_p$

$= \Sigma \rho_i \times m_i \dot{r}_i - (\cancel{\Sigma m_i \rho_i}) \times \dot{r}_p$

$= H_G$

O, G, P 점에서 관측한 G점에 대한 각운동량 값은 동일!

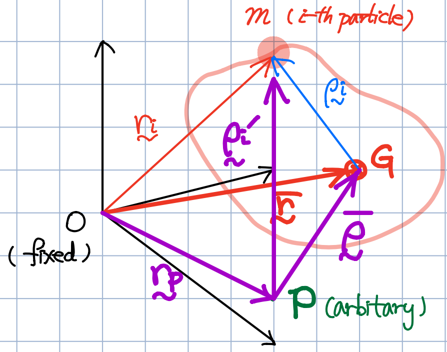
$H_G = \Sigma \rho_i \times m_i \dot{r}_i$

$\dot{H}_G = \Sigma \dot{\rho}_i \times m_i \dot{r}_i + \Sigma \rho_i \times m_i \ddot{r}_i$ ← $\cancel{\Sigma m_i \rho_i \times \ddot{r}_i}$

$= \Sigma \dot{\rho}_i \times m_i (\dot{\bar{r}} + \dot{\rho}_i)$

$= \cancel{\Sigma \dot{\rho}_i \times m_i \dot{\bar{r}}} + \cancel{\Sigma \dot{\rho}_i \times m_i \dot{\rho}_i}$

$\cancel{\Sigma m_i \dot{\rho}_i} \times \dot{\bar{r}}$ $\quad$ $\dot{\rho}_i // \dot{\rho}_i$



☆점에서 관측한 △에 대한 각운동량

☆	△	Answer
O	O	$\mathbf{H}_O = \mathbf{r}_i \times m_i\dot{\mathbf{r}}_i$
O	G	$\mathbf{H}_G = \boldsymbol{\rho}_i \times m_i\dot{\mathbf{r}}_i$
G	G	$(\mathbf{H}_G)_{rel\,G} = \boldsymbol{\rho}_i \times m_i\dot{\boldsymbol{\rho}}_i$
P	G	$(\mathbf{H}_G)_{rel\,P} = \boldsymbol{\rho}_i \times m_i\dot{\boldsymbol{\rho}}_i'$
O	P	$\mathbf{H}_P = \boldsymbol{\rho}_i' \times m_i\dot{\mathbf{r}}_i$
P	P	$(\mathbf{H}_P)_{rel\,P} = \boldsymbol{\rho}_i' \times m_i\dot{\boldsymbol{\rho}}_i'$

3. P점에 대한 각운동량

$$\mathbf{H}_P = \Sigma\boldsymbol{\rho}_i' \times m_i\dot{\mathbf{r}}_i = \Sigma(\bar{\boldsymbol{\rho}}+\boldsymbol{\rho}_i)\times m_i\dot{\mathbf{r}}_i$$

$$= \Sigma\bar{\boldsymbol{\rho}}\times m_i\dot{\mathbf{r}}_i + \underbrace{\Sigma\boldsymbol{\rho}_i\times m_i\dot{\mathbf{r}}_i}_{\mathbf{H}_G} \quad\longleftarrow\quad \Sigma m_i\dot{\mathbf{r}}_i = \Sigma m_i(\dot{\bar{\mathbf{r}}}+\dot{\boldsymbol{\rho}}_i)$$

$$= \Sigma m_i\dot{\bar{\mathbf{r}}} + \cancel{\Sigma m_i\dot{\boldsymbol{\rho}}_i}$$

$$= (\Sigma m_i)\dot{\bar{\mathbf{r}}}$$

$$= M\dot{\bar{\mathbf{r}}}$$

$$= \mathbf{H}_G + \Sigma\bar{\boldsymbol{\rho}}\times M\dot{\bar{\mathbf{r}}}$$

$$(\mathbf{H}_P)_{rel\,P} = \Sigma\boldsymbol{\rho}_i'\times m_i\dot{\boldsymbol{\rho}}_i' = \Sigma(\bar{\boldsymbol{\rho}}+\boldsymbol{\rho}_i)\times m_i\dot{\boldsymbol{\rho}}_i'$$

$$= \Sigma\bar{\boldsymbol{\rho}}\times m_i\dot{\boldsymbol{\rho}}_i' + \Sigma\boldsymbol{\rho}_i\times m_i\dot{\boldsymbol{\rho}}_i'$$

$$= \bar{\boldsymbol{\rho}}\times\Sigma m_i(\dot{\bar{\boldsymbol{\rho}}}+\dot{\boldsymbol{\rho}}_i) \qquad = (\mathbf{H}_G)_{rel\,P}$$

$$= \bar{\boldsymbol{\rho}}\times(\Sigma m_i)\dot{\bar{\boldsymbol{\rho}}} + \bar{\boldsymbol{\rho}}\times\cancel{\Sigma m_i\dot{\boldsymbol{\rho}}_i}$$

$$= \bar{\boldsymbol{\rho}}\times M\dot{\bar{\boldsymbol{\rho}}}$$

$$= (\mathbf{H}_G)_{rel\,P} + \bar{\boldsymbol{\rho}}\times M\dot{\bar{\boldsymbol{\rho}}}$$

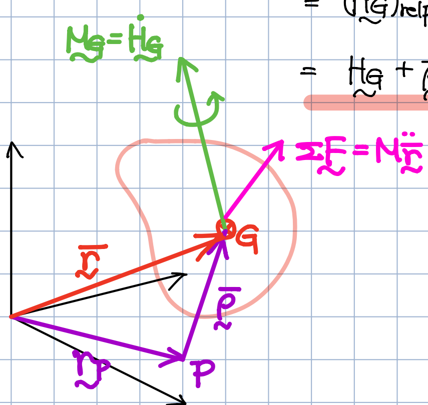
$$= \mathbf{H}_G + \bar{\boldsymbol{\rho}}\times M\dot{\bar{\boldsymbol{\rho}}}$$

$$(\mathbf{H}_P)_{rel\,P} = \mathbf{H}_G + \bar{\boldsymbol{\rho}}\times M\dot{\bar{\boldsymbol{\rho}}}$$

$$\mathbf{H}_G = (\mathbf{H}_P)_{rel\,P} - \bar{\boldsymbol{\rho}}\times M\dot{\bar{\boldsymbol{\rho}}} \quad\leftarrow\quad \frac{d}{dt}(\bar{\boldsymbol{\rho}}\times M\dot{\bar{\boldsymbol{\rho}}})$$

$$= \cancel{\dot{\bar{\boldsymbol{\rho}}}\times M\dot{\bar{\boldsymbol{\rho}}}} + \bar{\boldsymbol{\rho}}\times M\ddot{\bar{\boldsymbol{\rho}}}$$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_P)_{rel\,P} - \bar{\boldsymbol{\rho}}\times M\ddot{\bar{\boldsymbol{\rho}}}$$



$$\Sigma\mathbf{M}_P = \Sigma\mathbf{M}_G + \bar{\boldsymbol{\rho}}\times\Sigma\mathbf{F}$$

$$= \dot{\mathbf{H}}_G + \bar{\boldsymbol{\rho}}\times\Sigma\mathbf{F}$$

$$= (\dot{\mathbf{H}}_P)_{rel\,P} - \bar{\boldsymbol{\rho}}\times M\ddot{\bar{\boldsymbol{\rho}}} + \bar{\boldsymbol{\rho}}\times M\ddot{\bar{\mathbf{r}}}$$

O, G, P 점에서 관측한 G점에 대한 각운동량 값은 동일!

$$\mathbf{H}_G = \Sigma\boldsymbol{\rho}_i\times m_i\dot{\mathbf{r}}_i$$

$$\dot{\mathbf{H}}_G = \Sigma\dot{\boldsymbol{\rho}}_i\times m_i\dot{\mathbf{r}}_i + \Sigma\boldsymbol{\rho}_i\times m_i\ddot{\mathbf{r}}_i \quad\longleftarrow\quad \cancel{\Sigma m_i\boldsymbol{\rho}_i\times\ddot{\mathbf{r}}_i}$$

$$= \Sigma\dot{\boldsymbol{\rho}}_i\times m_i(\dot{\bar{\mathbf{r}}}+\dot{\boldsymbol{\rho}}_i)$$

$$= \cancel{\Sigma\dot{\boldsymbol{\rho}}_i\times m_i\dot{\bar{\mathbf{r}}}} + \cancel{\Sigma\dot{\boldsymbol{\rho}}_i\times m_i\dot{\boldsymbol{\rho}}_i}$$

$\cancel{\Sigma m_i\dot{\boldsymbol{\rho}}_i}\times\dot{\bar{\mathbf{r}}}$ $\qquad$ $\dot{\boldsymbol{\rho}}_i \parallel \dot{\boldsymbol{\rho}}_i$

$$\dot{\mathbf{H}}_G = \Sigma\boldsymbol{\rho}_i\times m_i\ddot{\mathbf{r}}_i = \Sigma\boldsymbol{\rho}_i\times\mathbf{f}_i = \Sigma\mathbf{M}_G$$

$$\therefore\ \Sigma\mathbf{M}_P = (\dot{\mathbf{H}}_P)_{rel\,P} + \bar{\boldsymbol{\rho}}\times M(\underbrace{\ddot{\bar{\mathbf{r}}}-\ddot{\bar{\boldsymbol{\rho}}}}_{=\ddot{\mathbf{r}}_P})$$

$$\boxed{\therefore\ \Sigma\mathbf{M}_P = (\dot{\mathbf{H}}_P)_{rel\,P} + \bar{\boldsymbol{\rho}}\times M\ddot{\mathbf{r}}_P}$$

The resultants of the external forces acting on the system expressed in terms of the resultant forces ($=\Sigma\mathbf{F}$) through G and the corresponding couple $\Sigma\mathbf{M}_G$.

$$\boxed{\therefore\ \Sigma \underset{\sim}{M}_P = (\dot{\underset{\sim}{H}}_P)_{rel\,P} + \underset{\sim}{\bar{\rho}} \times M \ddot{\underset{\sim}{r}}_P}$$

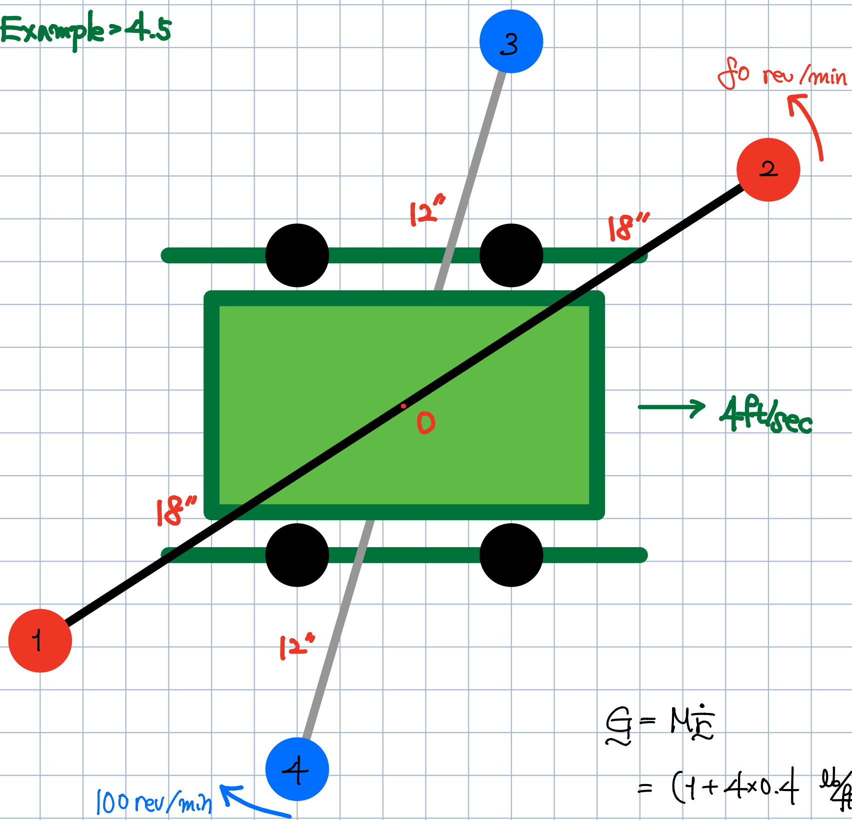
$\Sigma \underset{\sim}{M}_P = (\dot{\underset{\sim}{H}}_P)_{rel\,P}$ if,

① $\ddot{\underset{\sim}{r}}_P = \underset{\sim}{0}$ (관성좌표계) $\longrightarrow \Sigma \underset{\sim}{M}_O = \dot{\underset{\sim}{H}}_O$

② $\underset{\sim}{\bar{\rho}} = \underset{\sim}{0}$ (P점 = G점) $\longrightarrow \Sigma \underset{\sim}{M}_G = \dot{\underset{\sim}{H}}_G$

③ $\underset{\sim}{\bar{\rho}} \parallel \ddot{\underset{\sim}{r}}_P \longrightarrow \Sigma \underset{\sim}{M}_P = (\dot{\underset{\sim}{H}}_P)_{rel\,P}$.

Example > 4.5



32.2 lb carriage A moves with a speed of 4 ft/sec and carries two assemblies of balls and light rods which rotate about a shaft at O.

Each of the four balls weighs 3.22 lb

Determine $\underset{\sim}{G}$, $\underset{\sim}{H}_O$ = ?

$$\boxed{\begin{aligned} \underset{\sim}{G} &= M\dot{\underset{\sim}{\bar{r}}} \\ \underset{\sim}{H}_G &= \Sigma \underset{\sim}{\rho}_i \times m_i \dot{\underset{\sim}{r}}_i \end{aligned}}$$

$$\begin{aligned} \underset{\sim}{G} &= M\dot{\underset{\sim}{\bar{r}}} \\ &= (1 + 4 \times 0.4\ \text{lb}/\tfrac{ft}{s^2}) \cdot (4\,ft/s) \\ &= \underset{\sim\sim\sim}{5.6\ lb\text{-}sec}\ \text{Ans.} \end{aligned}$$

$$\begin{aligned} \underset{\sim}{H}_O = \underset{\sim}{H}_G &= \Sigma \underset{\sim}{\rho}_i \times m_i \dot{\underset{\sim}{r}}_i \\ &= \Sigma \rho_i m_i \dot{r}_i \sin 90^\circ \\ &= \left\{ (18\,\cancel{in}) \frac{1\,ft}{12\,\cancel{in}} \cdot (0.1\ \text{lb}/(\cancel{ft}/s^2)) \cdot (12.57\ \cancel{ft}/\cancel{s}) \right\} \times 2 \\ &\quad + \left\{ (12/12) \cdot (0.1) \cdot (-10.47) \right\} \times 2 \\ &= \underset{\sim\sim\sim}{1.676\ ft\text{-}lb\text{-}sec}\ \text{Ans.} \end{aligned}$$