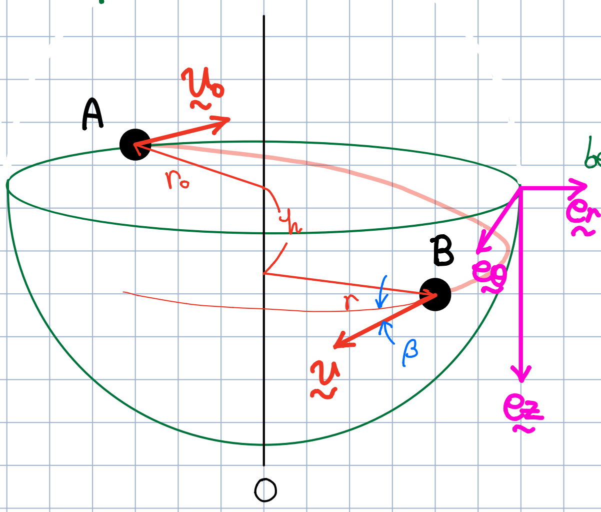
Example > 3.27

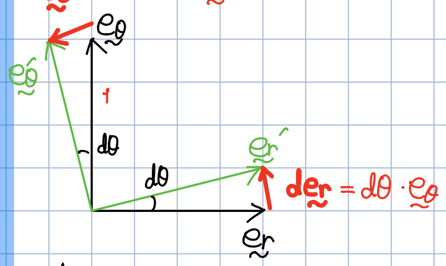


A small mass particle is given an initial velocity $\mathbf{u}_0$ at r_0.

As the particle slides past point B, a distance h below A and a distance r from the vertical centerline, its velocity $\mathbf{u}$ makes angle β with the horizontal tangent to the bowl through B.

Determine β.

$d\mathbf{e}_\theta = d\theta\,(-\mathbf{e}_r)$



$d\mathbf{e}_r/dt = d\theta \cdot \mathbf{e}_\theta/dt$ $\qquad$ $d\mathbf{e}_\theta/dt = d\theta\, -\mathbf{e}_r/dt$

$\dot{\mathbf{e}}_r = \mathbf{e}_\theta \cdot \dot\theta$ $\qquad$ $\dot{\mathbf{e}}_\theta = -\mathbf{e}_r\dot\theta$

1. Coordinate $\longrightarrow$ (r,θ,z) coord.

2. Vectors

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{k}$$

$$\dot{\mathbf{x}} = \dot r\mathbf{e}_r + r\underbrace{\dot{\mathbf{e}}_r}_{=\dot\theta\mathbf{e}_\theta} + \dot z\mathbf{k} + \cancel{z\dot{\mathbf{k}}}$$

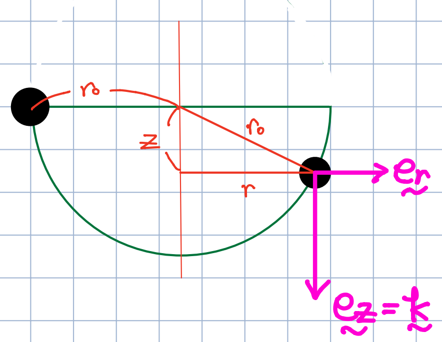
$$= \dot r\mathbf{e}_r + r\dot\theta\mathbf{e}_\theta + \dot z\mathbf{k}$$

$$\ddot{\mathbf{r}} = \ddot r\mathbf{e}_r + \dot r\underbrace{\dot{\mathbf{e}}_r}_{=\dot\theta\mathbf{e}_\theta} + \dot r\dot\theta\mathbf{e}_\theta + r\ddot\theta\mathbf{e}_\theta + r\dot\theta\underbrace{\dot{\mathbf{e}}_\theta}_{=-\mathbf{e}_r\dot\theta} + \ddot z\mathbf{k} + \cancel{\dot z\dot{\mathbf{k}}}$$

$$= \ddot r\mathbf{e}_r + 2\dot r\dot\theta\mathbf{e}_\theta + r\ddot\theta\mathbf{e}_\theta - r\dot\theta^2\mathbf{e}_r + \ddot z\mathbf{k}$$

$$= (\ddot r - r\dot\theta^2)\mathbf{e}_r + (2\dot r\dot\theta + r\ddot\theta)\mathbf{e}_\theta + \ddot z\mathbf{k}$$

3. Constraints



constrained : $r_0^2 = r^2 + z^2$

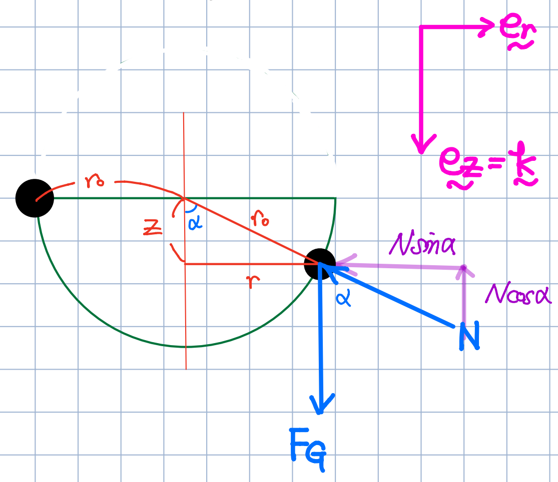
$$r^2 + z^2 = r_0^2$$

$$2r\dot r + 2z\dot z = 2r_0\cancel{\dot r_0}$$

$$r\dot r + z\dot z = 0$$

$$\dot r\dot r + r\ddot r + \dot z\dot z + z\ddot z = 0$$

4. Free Body Diagram



$$N = (N\sin\alpha)(-\underset{\sim}{e_r}) + (N\cos\alpha)\cdot(-\underset{\sim}{k})$$
$$= -N\sin\alpha\,\underset{\sim}{e_r} - N\cos\alpha\,\underset{\sim}{k}$$

$$F_G = (mg)\,\underset{\sim}{k}$$

5. Equation of Motion

$$\Sigma \underset{\sim}{F} = m\ddot{\underset{\sim}{r}} \quad \longleftarrow \quad \ddot{\underset{\sim}{r}} = (\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underset{\sim}{e_\theta} + \ddot{z}\underset{\sim}{k}$$

$$= m(\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e_r} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\underset{\sim}{e_\theta} + m\ddot{z}\underset{\sim}{k}$$

$$= -N\sin\alpha\,\underset{\sim}{e_r} - N\cos\alpha\,\underset{\sim}{k} + mg\,\underset{\sim}{k}$$

1. $m(\ddot{r} - r\dot{\theta}^2) = -N\sin\alpha \quad \longleftarrow \quad \sin\alpha = r/r_0$

   $= -N \cdot r/r_0$

   < unknowns : 4개
   식 : 4개.

2. $m\ddot{z} = mg - N\cos\alpha = mg - N\,z/r_0$

3. $m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$

4. constrained : $r_0^2 = z^2 + r^2$

from 3 : $\frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \longleftarrow \quad \frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$

$$r^2\dot{\theta} = \text{constant}$$

6. Calculation

integration of E.O.M. with respect to path.

$$\int_A^B m\ddot{\underset{\sim}{r}}\,d\underset{\sim}{r} = \int_A^B \underset{\sim}{F_G}\,d\underset{\sim}{r} + \cancel{\int_A^B \underset{\sim}{N}\,d\underset{\sim}{r}} \quad \longleftarrow \quad \begin{aligned} F_G &= mg\,\underset{\sim}{k} \\ d\underset{\sim}{r} &= dr\,\underset{\sim}{e_r} + r\,d\theta\,\underset{\sim}{e_\theta} + dz\,\underset{\sim}{k} \end{aligned}$$

$$= \int_A^B (mg\,\underset{\sim}{k})\cdot(\cancel{dr\,\underset{\sim}{e_r}} + \cancel{r\,d\theta\,\underset{\sim}{e_\theta}} + dz\,\underset{\sim}{k})$$

$$= \int_A^B mg\,dz = mgh$$

$*$ $v = ds/dt$ $\quad$ $a = dv/dt$

$\downarrow$ $\quad \longleftarrow \quad dt = dv/a$

$v = ds/(dv/a) = a\,ds/dv$

$\therefore\ a\,ds = v\,dv$

$\ddot{r}\,dr = v\,dv$

$\therefore \displaystyle\int_A^B m\ddot{\mathbf{r}}\,d\mathbf{r} = \int_A^B mv\,dv$

$= \frac{1}{2}m(v_B^2 - v_A^2)$

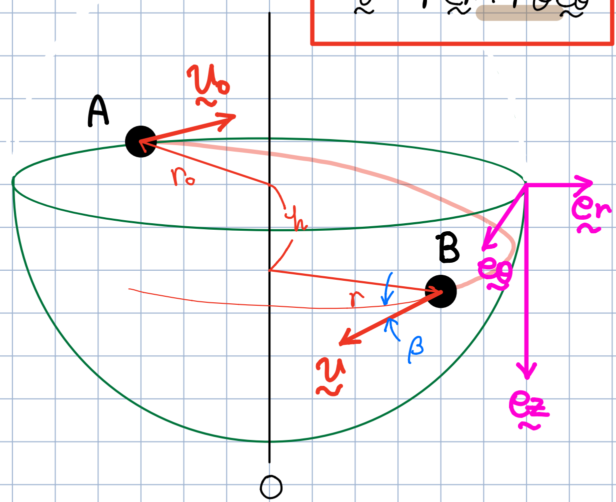
$\frac{1}{2}\cancel{m}v_B^2 - \frac{1}{2}\cancel{m}v_A^2 = \cancel{m}gh$

$v_B = \sqrt{2gh + v_A^2}$

$\mathbf{r} = r\mathbf{e}_r$

$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \longleftarrow \dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$

$\boxed{\therefore\ \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta}$



$\mathbf{e}_\theta \cdot \mathbf{v}_B = \underbrace{|\mathbf{e}_\theta|}_{=1}\,|\mathbf{v}_B|\cos\beta$

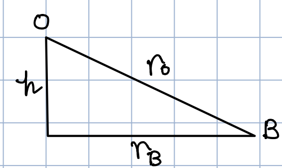
$= \sqrt{2gh + v_A^2}\cos\beta$

$= (r\dot{\theta})_B$

$*$ $r^2\dot{\theta} = \text{constant}$

$(r^2\dot{\theta})_B = (r^2\dot{\theta})_A = (r_0^2\dot{\theta})_A$

$(r\dot{\theta})_B = \frac{1}{r_B}(r_0^2\dot{\theta})_A = \frac{1}{r_B}(r_0 v_A) \longleftarrow v_A = r_0\dot{\theta}$



$\sqrt{2gh + v_A^2}\cos\beta = \frac{1}{r_B}(r_0 v_A)$

$\cos\beta = \dfrac{\frac{1}{r_B}(r_0 v_A)}{\sqrt{2gh + v_A^2}} \longleftarrow r_B = \sqrt{r_0^2 - h^2}$

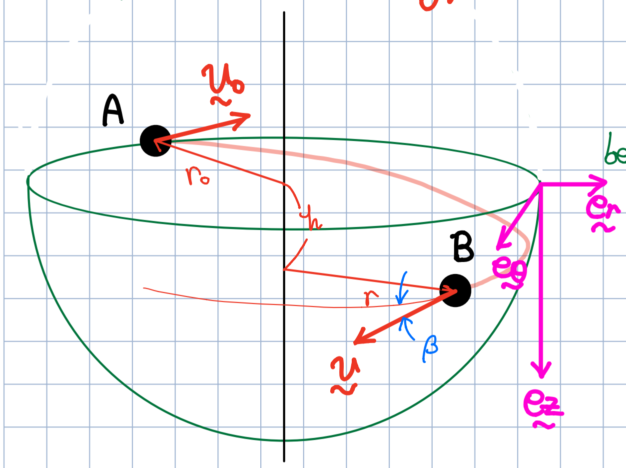
$= \dfrac{r_0 v_A}{\sqrt{r_0^2 - h^2}\cdot\sqrt{2gh + v_A^2}}$

$= \sqrt{r_0^2(1 - (h/r_0)^2)}$ $\quad = \sqrt{v_A^2\left(1 + \frac{2gh}{v_A^2}\right)}$

$\beta = \cos^{-1}\left(\dfrac{1}{\sqrt{1 - (h/r_0)^2}\cdot\sqrt{v_0^2\left(1 + \frac{2gh}{v_0^2}\right)}}\right) \longleftarrow v_A = v_0$

Ans.

Example 3.27 (Energy)



A small mass particle is given an initial velocity $\underset{\sim}{v_0}$ at r_0.

As the particle slides past point B, a distance h below A and a distance r from the vertical centerline, its velocity $\underset{\sim}{v}$ makes angle β with the horizontal tangent to the bowl through B.

Determine β.

* $(H_O)_1 = (H_O)_2$ conservation of angular momentum

$\underset{\sim}{H_O} = \underset{\sim}{r} \times m\underset{\sim}{v}$

$= rmv\sin 90^\circ$

$= rmv$

$(H_O)_1 = r_0 m v_0$ $\longrightarrow$ $\cancel{m}v_0 r_0 = \cancel{m} v r\cos\beta$ ∴ $v_0 r_0 = \sqrt{v_0^2+2gh}\cdot\sqrt{r_0^2-h^2}\cdot\cos\beta$

$(H_O)_2 = rmv\cos\beta$

∴ $\beta = \cos^{-1}\left(\dfrac{1}{\sqrt{1+\left(\frac{2gh}{v_0}\right)}\cdot\sqrt{1-(h/r_0)^2}}\right)$ Ans.

* $E_1 = E_2$ conservation of energy

$T_1+V_1 = T_2+V_2$

$\frac{1}{2}mv_0^2 + mgh = \frac{1}{2}mv^2 + 0$

∴ $v = \sqrt{v_0^2+2gh}$

* $r^2 = r_0^2 - h^2$