

자동제어 한방에 끝내기

3. Laplace transforms

목차

Time domain vs S-domain

- 라플라스 변환을 왜 사용하는가?

Laplace transform

- Final Value Problem
- Poles, Zeros의 영향

Transfer functions, Time domain characteristics

- 라플라스 변환 분석과 시간 해석
- MATLAB을 이용한 해석법 – 실습

안정성 분석

- Routh's stability criterion을 이용한 분석

라플라스 역변환 – partial fraction

- Partial fraction

- Given $F(s) = \frac{b_1 s^m + b_2 s^{m-1} + \dots + b_{m+1}}{s^n + a_1 s^{n-1} + \dots + a_n} = K \frac{\prod_{i=1}^m (s - z_i)^{\text{zero}}}{\prod_{i=1}^n (s - p_i)^{\text{pole}}}$
- $a_1, \dots, a_n, b_1, \dots, b_{m+1} \in \mathbb{R}, \underbrace{(s^2 + s + 1)}_{-\frac{1}{2} \pm \frac{\sqrt{3}}{2}j} z_1, \dots, z_m, p_1, \dots, p_n \in \mathbb{C}$
- $\frac{1}{s - p_1} + \frac{1}{s - p_2} + \dots$ 의 방식으로 어떻게 정리할 것인가?

라플라스 역변환 – partial fraction

Cover-up method

$$\gamma(s) = \frac{1}{(s-a)} + \frac{1}{(s-b)} + \sim$$

$$\gamma(a) = \infty$$

$$(s-a)\gamma(s) = 1 + \frac{s-a}{s-b} + \sim$$

• Partial fraction

$$\bullet H(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}, \quad U(s) = \frac{1}{s}, \quad u(t) = 1(t), \quad y(t) = ?$$

$$\gamma(s) = H(s)U(s) = \frac{(s+1)(s+2)}{s(s+3)(s+4)} = \frac{C_0}{s} + \frac{C_1}{s+3} + \frac{C_2}{s+4} = \frac{1}{6} \frac{1}{s} - \frac{2}{3} \frac{1}{s+3} + \frac{3}{2} \frac{1}{s+4}$$

$$C_0 = \left[s \cdot \frac{(s+1)(s+2)}{s(s+3)(s+4)} \right]_{s=0} = \frac{1 \cdot 2}{3 \cdot 4} = \frac{1}{6}$$

$$C_1 = \left[(s+3) \frac{(s+1)(s+2)}{s(s+3)(s+4)} \right]_{s=-3} = \frac{-2 \cdot -1}{-3 \cdot 1} = -\frac{2}{3}$$

$$C_2 = \frac{-3 \cdot -2}{-4 \cdot -1} = \frac{3}{2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{1}{6} 1(t) - \frac{2}{3} e^{-3t} 1(t) + \frac{3}{2} e^{-4t} 1(t)$$

$$\bullet H(s) = \frac{1}{(s+2)^2}$$

$$\gamma(s) = H(s)U(s) = \frac{1}{s(s+2)^2} = \frac{C_0}{s} + \frac{C_1}{s+2} + \frac{C_2}{(s+2)^2} = \frac{1}{4} \frac{1}{s} - \frac{1}{4} \frac{1}{s+2} - \frac{1}{2} \frac{1}{(s+2)^2}$$

$$C_0 = \frac{1}{4}, \quad C_2 = -\frac{1}{2}$$

$$C_0 s^2 + C_1 s^2 = 0 s^2 \Rightarrow C_0 + C_1 = 0 \rightarrow C_1 = -C_0 = -\frac{1}{4}$$

$$\frac{1}{s^2} = \frac{1}{s} \cdot \frac{1}{s} = \int \frac{1}{s} \frac{1}{s} ds = \int 1(t) dt = t$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{1}{4} 1(t) - \frac{1}{4} e^{-2t} 1(t) - \frac{1}{2} t e^{-2t} 1(t)$$

라플라스 역변환 – partial fraction

• Partial fraction

$$H(s) = \frac{2s^2+s+1}{s^3-1}, u(t) = \delta(t), y(t) = ?$$

$$\gamma(s) = H(s) = \frac{2s^2+s+1}{(s-1)(s^2+s+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+s+1}$$

$$A = \frac{4}{3} \quad 1 = A - C \rightarrow C = \frac{1}{3}$$

$$2 = A + B \rightarrow B = \frac{2}{3}$$

$$= \frac{4}{3} \cdot \frac{1}{(s-1)} + \frac{2}{3} \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{4}{3} e^t 1(t) + \frac{2}{3} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) 1(t)$$

$$H(s) = \tan^{-1}\left(\frac{1}{s}\right)$$

$$(\tan^{-1}x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

$$\tan^{-1}x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\tan^{-1}\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3!}\frac{1}{s^3} + \frac{1}{5!}\frac{1}{s^5} - \dots \xrightarrow{\mathcal{L}^{-1}} 1(t) \left\{ 1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \dots \right\} = \frac{\sin t}{t} 1(t)$$

$$t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \dots \rightarrow \sin t$$

라플라스 역변환 – partial fraction

- Partial fraction

- $H(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}, u(t) = 1(t), y(t) = ?$

- MATLAB -> `[coeff,pole,k]=residue(poly([-1,-2]),poly([0,-3,-4]));`

- $H(s) = \frac{2s^2+s+1}{s^3-1}, u(t) = \delta(t), y(t) = ?$

- MATLAB -> `[coeff,pole,k]=residue([2 1 1],[1 0 0 -1]);`

라플라스 역변환 – partial fraction

- Partial fraction

The image shows the MATLAB R2022a interface with the following components:

- Toolbar:** Includes icons for file operations (new, open, save, print), editing (undo, redo, find), and execution (run, stop, clear).
- Current Folder:** Displays the path `C:\Users\aldri\Documents\MATLAB`.
- Editor:** Contains two lines of MATLAB code:

```
1 [coeff,pole,k]=residue(poly([-1,-2]),poly([0,-3,-4]))  
2 [coeff,pole,k]=residue([2 1 1],[1 0 0 -1])
```
- Command Window:** Displays the results of the partial fraction decomposition:

```
coeff = 3x1  
    1.5000  
   -0.6667  
    0.1667  
  
pole = 3x1  
    -4  
    -3  
     0  
  
k =  
  
[]  
  
coeff = 3x1 complex  
    0.3333 - 0.0000i  
    0.3333 + 0.0000i  
    1.3333 + 0.0000i  
  
pole = 3x1 complex  
   -0.5000 + 0.8660i  
   -0.5000 - 0.8660i  
    1.0000 + 0.0000i  
  
k =  
  
[]
```
- Handwritten Equation:** The partial fraction decomposition is shown as:
$$\frac{3}{(s+4)} + \frac{-\frac{2}{3}}{(s+3)} + \frac{1}{s}$$

Final value Theorem

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = \int_0^{\infty} e^{-st} \frac{dy}{dt} dt$$

$$\lim_{s \rightarrow 0} \{sY(s) - y(0)\} = \lim_{s \rightarrow 0} \int_0^{\infty} \cancel{e^{-st}} \frac{dy}{dt} dt = \lim_{t \rightarrow \infty} \{y(t) - y(0)\}$$

Final value Theorem

$$\lim_{t \rightarrow \infty} \underbrace{y(t)}_{\rightarrow \infty} = \lim_{s \rightarrow 0} sY(s)$$

- FVT의 의의
 - '최종적으로' 도달하는 값을 획득
- 안정성 분석

Pole의 위치

- Pole?

- $$H(s) = \frac{1}{s+1/\tau} \leftrightarrow h(t) = e^{-\frac{t}{\tau}} 1(t)$$

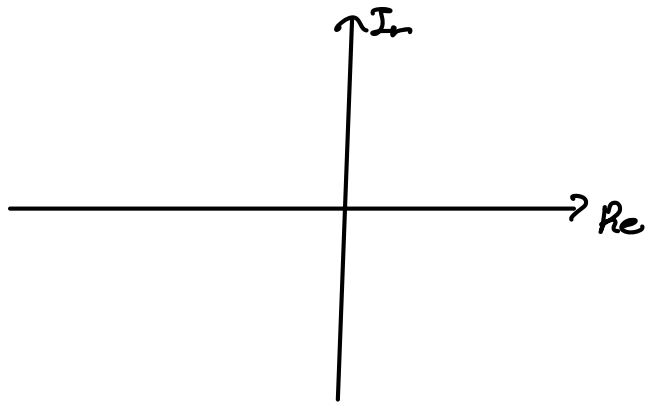
$$\frac{1}{s+a} \xrightarrow{\mathcal{L}^{-1}} e^{-at} 1(t)$$

Pole과 안정성

- Pole?

$$\frac{1}{s+a} \rightarrow \text{pole} : -a$$

$$\bullet H(s) = \frac{1}{\underbrace{s + 1/\tau}_{\in \text{LHP}}} \leftrightarrow \underline{h(t) = e^{-\frac{t}{\tau}} 1(t)}$$



Pole : LHP Left-Half-plane

2차 시스템의 라플라스 변환

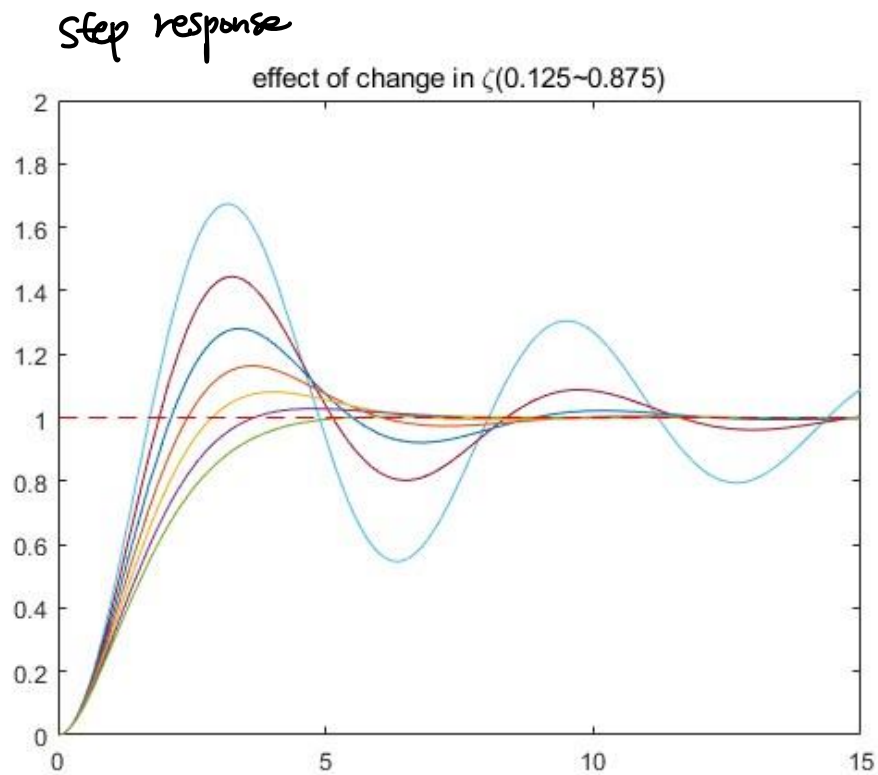
- 시스템의 전달함수가 두개의 켤레복소근을 가지면

- $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \leftrightarrow h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin \omega_d t 1(t)$

- $\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\begin{aligned}
 H(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + \omega_n^2 - \zeta^2\omega_n^2} = \frac{\left(\sqrt{1-\zeta^2}\omega_n\right)^2}{\underbrace{(s + \frac{\zeta\omega_n}{2})^2}_{\sigma} + \underbrace{\left(\sqrt{1-\zeta^2}\omega_n\right)^2}_{\omega_d^2}} \cdot \frac{1}{1-\zeta^2} \\
 &= \frac{\omega_d}{(s + \sigma)^2 + \omega_d^2} \cdot \frac{\omega_d}{1-\zeta^2} = \frac{\omega_n}{\sqrt{1-\zeta^2}} \cdot \frac{\omega_d}{(s + \sigma)^2 + \omega_d^2} \\
 &\xrightarrow{\mathcal{L}^{-1}} \frac{\omega_n}{\sqrt{1-\zeta^2}} \cdot e^{-\sigma t} \sin(\omega_d t) 1(t)
 \end{aligned}$$

2차 시스템의 라플라스 변환



```
syms s
f1 = ilaplace(1/(s^2+0.25*s+1)/s);
f2 = ilaplace(1/(s^2+0.5*s+1)/s);
f3 = ilaplace(1/(s^2+0.75*s+1)/s);
f4 = ilaplace(1/(s^2+1*s+1)/s);
f5 = ilaplace(1/(s^2+1.25*s+1)/s);
f6 = ilaplace(1/(s^2+1.5*s+1)/s);
f7 = ilaplace(1/(s^2+1.75*s+1)/s);
fplot(f1);
hold on
fplot(f2);
fplot(f3);
fplot(f4);
fplot(f5);
fplot(f6);
fplot(f7);
yline(1,'r--')
xlim([0 15])
title('effect of change in \zeta(0.125~0.875)')
hold off
```