자동제어 한방에 끝내기

3. Laplace transforms

목차

Time domain vs S-domain

- 라플라스 변환을 왜 사용하는가?

Laplace transform

- Final Value Problem
- Poles, Zeros의 영향

Transfer functions, Time domain characteristics

- 라플라스 변환 분석과 시간 해석
- MATLAB을 이용한 해석법 – 실습

안정성 분석

- Routh's stability criterion을 이용한 분석

라플라스 역변환 – partial fraction

- Partial fraction
  - Given $F(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = K \frac{\prod_{i=1}^{m}(s - z_i)}{\sum_{i=1}^{n}(s - p_i)}$ zero pole
  - $a_1, \dots, a_n, b_1, \dots, b_{m+1} \in \mathbb{R}$, (s^2+s+1) $z_1, \dots z_m, p_1, \dots, p_n \in \mathbb{Z}$

    $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$
  - $\frac{1}{s-p_1} + \frac{1}{s-p_2} + \cdots$ 의 방식으로 어떻게 정리할 것인가?

라플라스 역변환 – partial fraction

Cover-up method

$$Y(s) = \frac{1}{(s-a)} + \frac{1}{(s-b)} + \sim$$

$$Y(a) = \infty$$

$$(s-a)Y(s) = 1 + \frac{s-a}{s-b} + \sim$$

- Partial fraction
  - $H(s) = \dfrac{(s+1)(s+2)}{(s+3)(s+4)}$, $u(t) = 1(t)$, $y(t) = ?$

$$U(s) = \frac{1}{s}$$

$$Y(s) = H(s)U(s) = \frac{(s+1)(s+2)}{s(s+3)(s+4)} = \frac{C_0}{s} + \frac{C_1}{s+3} + \frac{C_2}{s+4} = \frac{1}{6}\frac{1}{s} - \frac{2}{3}\frac{1}{s+3} + \frac{3}{2}\frac{1}{s+4}$$

$$C_0 = \left[s \cdot \frac{(s+1)(s+2)}{s(s+3)(s+4)}\right]_{s=0} = \frac{1 \cdot 2}{3 \cdot 4} = \frac{1}{6} \qquad C_1 = \left[(s+3)\frac{(s+1)(s+2)}{s(s+3)(s+4)}\right]_{s=-3} = \frac{-2 \cdot -1}{-3 \cdot 1} = -\frac{2}{3}$$

$$C_2 = \frac{-3 \times -2}{-4 \times -1} = \frac{3}{2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{1}{6}1(t) - \frac{2}{3}e^{-3t}1(t) + \frac{3}{2}e^{-4t}1(t)$$

- $H(s) = \dfrac{1}{(s+2)^2}$

$$Y(s) = H(s)U(s) = \frac{1}{s(s+2)^2} = \underset{(s+2)^2}{\frac{C_0}{s}} + \underset{s(s+2)}{\frac{C_1}{s+2}} + \underset{s}{\frac{C_2}{(s+2)^2}} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{(s+2)} - \frac{1}{2}\frac{1}{(s+2)^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{1}{4}1(t) - \frac{1}{4}e^{-2t}1(t) - \frac{1}{2}te^{-2t}1(t)$$

$$C_0 = \frac{1}{4}, \quad C_2 = -\frac{1}{2}$$

$$C_0s^2 + C_1s^2 = 0s^2 \Rightarrow C_0 + C_1 = 0 \longrightarrow C_1 = -C_0 = -\frac{1}{4}$$

$$\frac{1}{s^2} = \underset{\int}{\frac{1}{s}} \cdot \underset{1(t)}{\frac{1}{s}} = \int 1(t)\,dt = t$$

라플라스 역변환 – partial fraction

- Partial fraction
  - $H(s) = \frac{2s^2+s+1}{s^3-1}$, $u(t) = \delta(t)$, $y(t) = ?$ $\quad U(s) = 1$

$$Y(s) = H(s) = \frac{2s^2+s+1}{(s-1)(s^2+s+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+s+1} = \frac{1}{(s-1)} \times \frac{4}{3} + \frac{\frac{2}{3}s + \frac{1}{3}}{s^2+s+1}$$

$$A = \frac{4}{3} \qquad 1 = A - C \rightarrow C = \frac{1}{3} \qquad 2 = A + B \rightarrow B = \frac{2}{3}$$

$$= \frac{4}{3} \cdot \frac{1}{(s-1)} + \frac{2}{3} \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \underbrace{\frac{3}{4}}_{\omega^2}}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{4}{3} e^{t} 1(t) + \frac{2}{3} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) 1(t)$$

  - $H(s) = \tan^{-1}\left(\frac{1}{s}\right)$

$$(\tan^{-1} x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots$$

$$\tan^{-1} x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\tan^{-1}\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3!}\frac{1}{s^3} + \frac{1}{5!}\frac{1}{s^5} - \cdots \xrightarrow{\mathcal{L}^{-1}} 1(t)\left\{1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \cdots\right\} = \frac{\sin t}{t} 1(t)$$

$$t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots \rightarrow \sin t$$

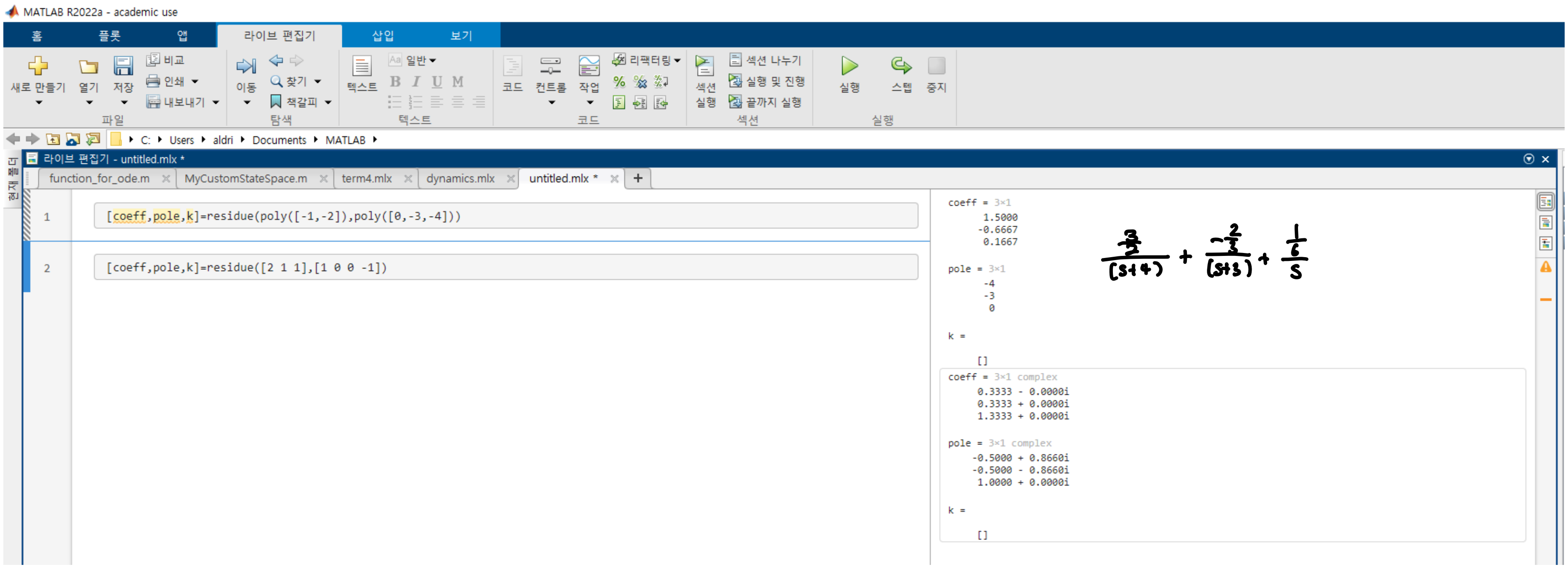
라플라스 역변환 – partial fraction

- Partial fraction
  - $H(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}, u(t) = 1(t), y(t) = ?$
  - MATLAB -> [coeff,pole,k]=residue(poly([-1,-2]),poly([0,-3,-4]));

  - $H(s) = \frac{2s^2+s+1}{s^3-1}, u(t) = \delta(t), y(t) = ?$
  - MATLAB -> [coeff,pole,k]=residue([2 1 1],[1 0 0 -1]);

라플라스 역변환 – partial fraction

- Partial fraction



Final value Theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = \int_0^\infty e^{-st}\frac{dy}{dt}\,dt$$

$$\lim_{s\to 0}\{sY(s) - y(0)\} = \lim_{s\to 0}\int_0^\infty \cancel{e^{-st}}\,\frac{dy}{dt}\,dt = \lim_{t\to\infty}\{y(t) - y(0)\}$$

Final value Theorem

$$\lim_{t\to\infty} \underbrace{y(t)}_{\to\infty} = \lim_{s\to 0} sY(s)$$

- FVT의 의의
  - '최종적으로' 도달하는 값을 획득

  - 안정성 분석

Pole의 위치

- Pole?
  - $H(s) = \dfrac{1}{s+1/\tau} \leftrightarrow h(t) = e^{-\frac{t}{\tau}}1(t)$

$$\frac{1}{s+a} \xrightarrow{\mathcal{L}^{-1}} e^{-at}1(t)$$

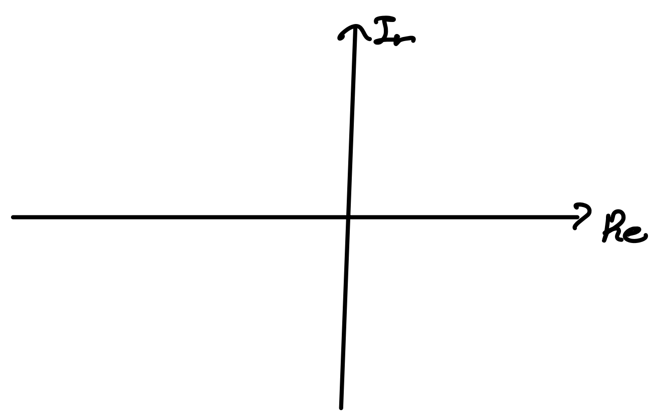
Pole과 안정성

- Pole?

$\frac{1}{s+a} \longrightarrow$ pole : $-a$

- $H(s) = \frac{1}{s+1/\tau} \leftrightarrow h(t) = e^{-\frac{t}{\tau}}1(t)$

$\in$ 복소수



Pole : $\boxed{\text{LHP}}$ Left-Half-Plane

2차 시스템의 라플라스 변환

- 시스템의 전달함수가 두개의 켤레복소근을 가지면
- $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \leftrightarrow h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin \omega_d t \, 1(t)$
  - $\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}$

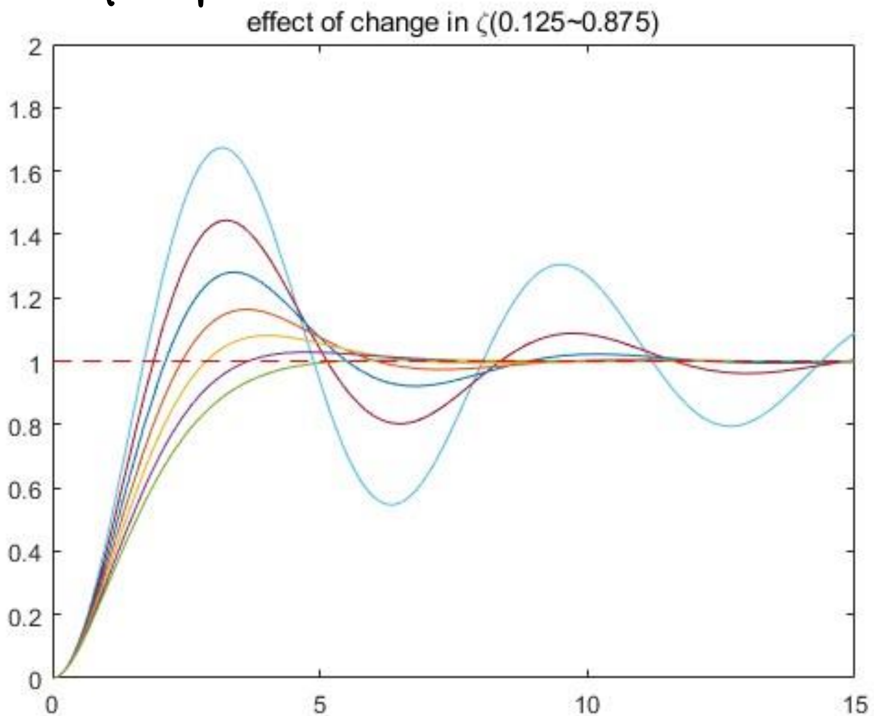
$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\zeta^2\omega_n^2)+\omega_n^2-\zeta^2\omega_n^2} = \frac{(\overbrace{\sqrt{1-\zeta^2}\,\omega_n}^{\omega_d})^2}{(s+\underbrace{\zeta\omega_n}_{\sigma})^2+(\underbrace{\sqrt{1-\zeta^2}\,\omega_n}_{\omega_d})^2} \cdot \frac{1}{1-\zeta^2}$$

$$= \frac{\omega_d}{(s+\sigma)^2+\omega_d^2} \cdot \frac{\overset{\sqrt{1-\zeta^2}\,\omega_n}{\omega_d}}{1-\zeta^2} = \frac{\omega_n}{\sqrt{1-\zeta^2}} \cdot \frac{\omega_d}{(s+\sigma)^2+\omega_d^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{\omega_n}{\sqrt{1-\zeta^2}} \cdot e^{-\sigma t} \sin(\omega_d t)\, 1(t)$$

2차 시스템의 라플라스 변환

step response




```
syms s
f1 = ilaplace(1/(s^2+0.25*s+1)/s);
f2 = ilaplace(1/(s^2+0.5*s+1)/s);
f3 = ilaplace(1/(s^2+0.75*s+1)/s);
f4 = ilaplace(1/(s^2+1*s+1)/s);
f5 = ilaplace(1/(s^2+1.25*s+1)/s);
f6 = ilaplace(1/(s^2+1.5*s+1)/s);
f7 = ilaplace(1/(s^2+1.75*s+1)/s);
fplot(f1);
hold on
fplot(f2);
fplot(f3);
fplot(f4);
fplot(f5);
fplot(f6);
fplot(f7);
yline(1,'r--')
xlim([0 15])
title('effect of change in \zeta(0.125~0.875)')
hold off
```