자동제어 한방에 끝내기

3. Laplace transforms

목차

Time domain vs S-domain

- 라플라스 변환을 왜 사용하는가?

Laplace transform

- Final Value Problem, Initial Value Problem
- Poles, Zeros의 영향

Transfer functions, Time domain characteristics

- 라플라스 변환 분석과 시간 해석
- MATLAB을 이용한 해석법 – 실습

안정성 분석

- Routh’s stability criterion을 이용한 분석

Time domain vs S-domain

- LTI(linear time-invariant) Systems
  - 선형 시불변 시스템 – 상수들이 시간에 대해 변하지 않는 **선형** 시스템
  - 선형 시스템 – 어떤 변수의 시간에 대한 미분만으로 이루어진 시스템
    - ex) $m\ddot{x} + c\dot{x} + kx \overset{x_a,\, x_b}{=} F(t)$, $L\frac{dI}{dt} + RI + \frac{1}{C}\int I dt = V(t)$
    - 반례) $m\ddot{x} + c\dot{x} + k_1 x + k_2 x^3 = F(t)$
  - 선형 시스템?

$$m\ddot{x}_a + c\dot{x}_a + kx_a = A$$
$$m\ddot{x}_b + c\dot{x}_b + kx_b = B$$
$$m\ddot{x}_a + m\ddot{x}_b + c\dot{x}_a + c\dot{x}_b + kx_a + kx_b = A+B$$

$$m\ddot{x}_a + c\dot{x}_a + k_1 x_a + k_2 x_a^3 = \alpha$$
$$m\ddot{x}_b + c\dot{x}_b + k_1 x_b + k_2 x_b^3 = \beta$$

$$\longrightarrow \alpha + \beta = m(\ddot{x_a + x_b}) + c(\dot{x_a + x_b}) + k_1(x_a + x_b) + k_2(x_a + x_b)^3$$

Time domain vs S-domain

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    - 반례) $m\ddot{x} + c\dot{x} + k_1x + k_2x^3 = F(t)$
  - 선형 시스템?
    - **Superposition!**
    - **Convolution!**

Superposition

- **선형 시스템의 가장 중요한 특징**

$$given \ \ m\ddot{x}_a + c\dot{x}_a + kx_a = A(t), m\ddot{x}_b + c\dot{x}_b + kx_b = B(t)$$

$$m\ddot{x}_c + c\dot{x}_c + kx_c = A(t) + B(t)$$

$$x_c = ?$$

Superposition

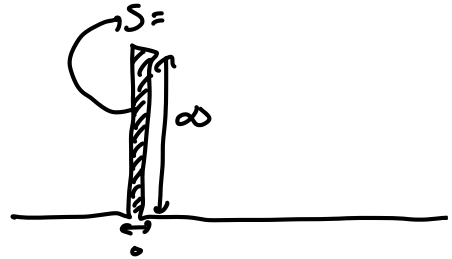
- **선형 시스템의 가장 중요한 특징**

$given\ \ m\ddot{x}_a + c\dot{x}_a + kx_a = A(t), m\ddot{x}_b + c\dot{x}_b + kx_b = B(t)$

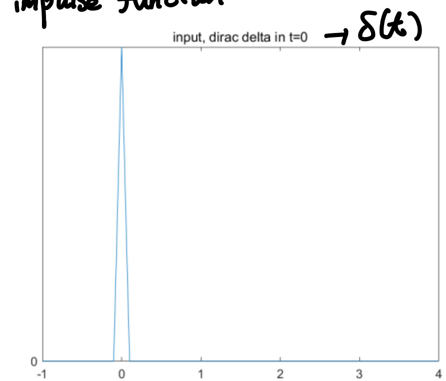
$$m\ddot{x}_c + c\dot{x}_c + kx_c = A(t) + B(t)$$

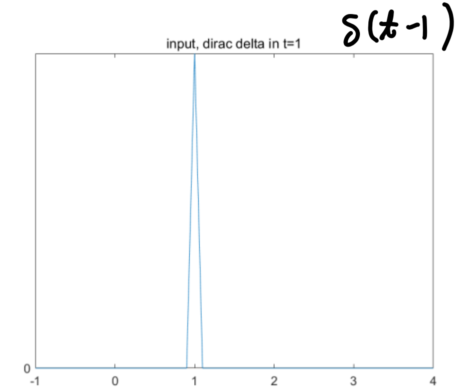
$$\boldsymbol{x_c = x_a + x_b}$$

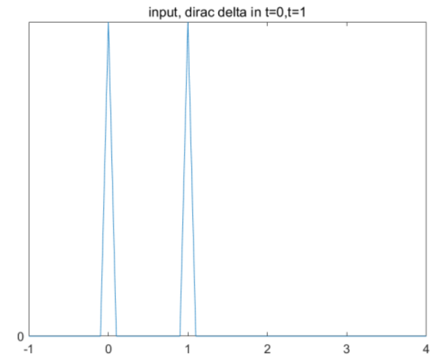
Superposition

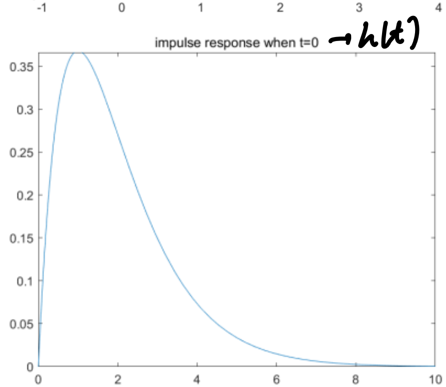


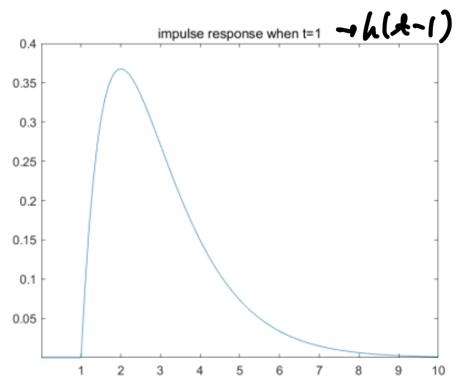
impulse function

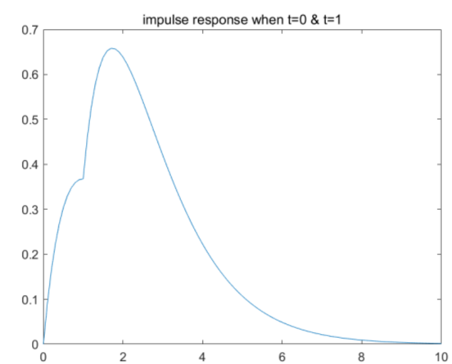












$+\alpha$: MATLAB Plotting code

```
s = tf('s');
impulse(1/(s+1)^2)
title('impulse response when t=0')
impulse(exp(-1*s)/(s+1)^2)
title('impulse response when t=1')
impulse(1/(s+1)^2+exp(-1*s)/(s+1)^2)
title('impulse response when t=0 & t=1')
```

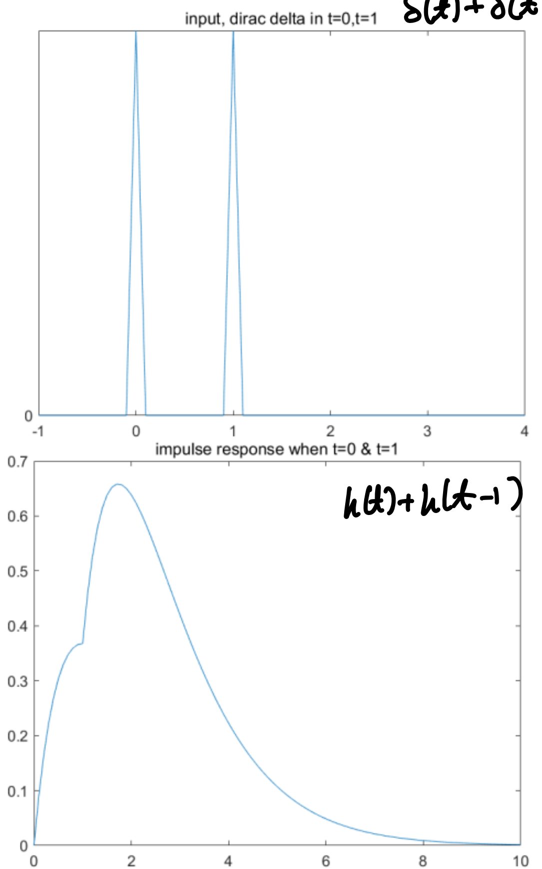
```
x = [-1:0.1:4];
p = zeros(51,1);
p(11)=1;
plot(x,p)
yticks([0])
title('input, dirac delta in t=0')

x = [-1:0.1:4];
p = zeros(51,1);
p(21)=1;
plot(x,p)
yticks([0])
title('input, dirac delta in t=1')

x = [-1:0.1:4];
p = zeros(51,1);
p(11)=1;
p(21)=1;
plot(x,p)
yticks([0])
title('input, dirac delta in t=0,t=1')
```

Time domain vs S-domain

$\delta(t) + \delta(t-1)$



- Given $\delta(t) \ \rightarrow h(t)$
  - $\delta(t-\tau) \ \rightarrow h(t-\tau)$

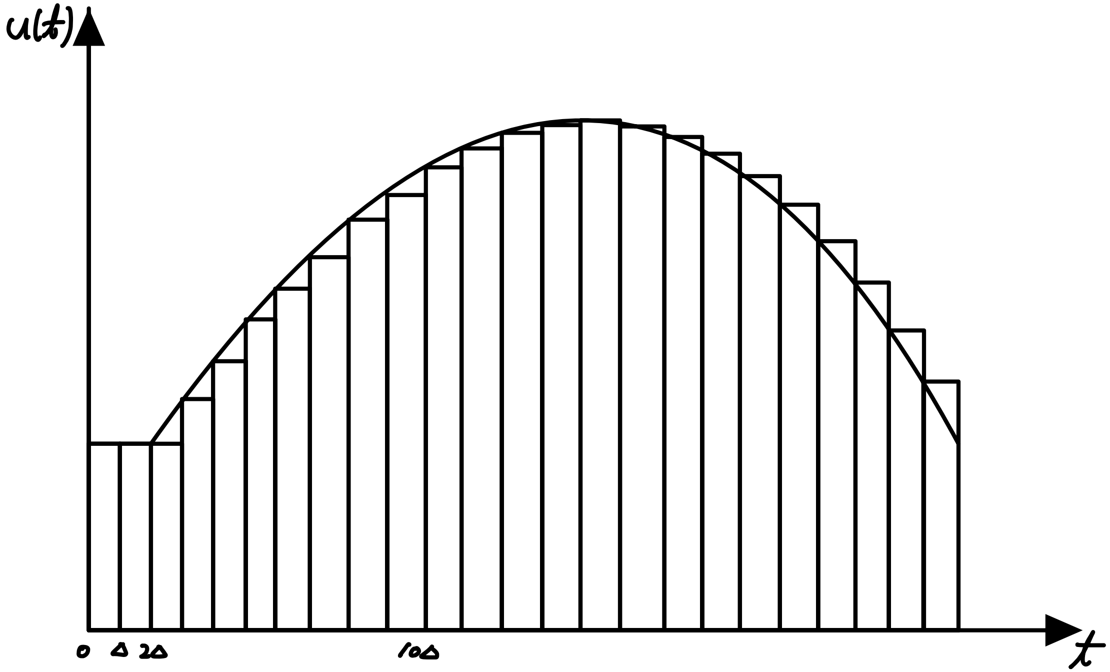
- 입력이$\delta(t) + \delta(t-\tau)$라면?

  $h(t) + h(t-\tau)$

- 입력이 $u_1\delta(t) + \ u_2\delta(t-\tau)$라면?

  $u_1 h(t) + u_2 h(t-\tau)$

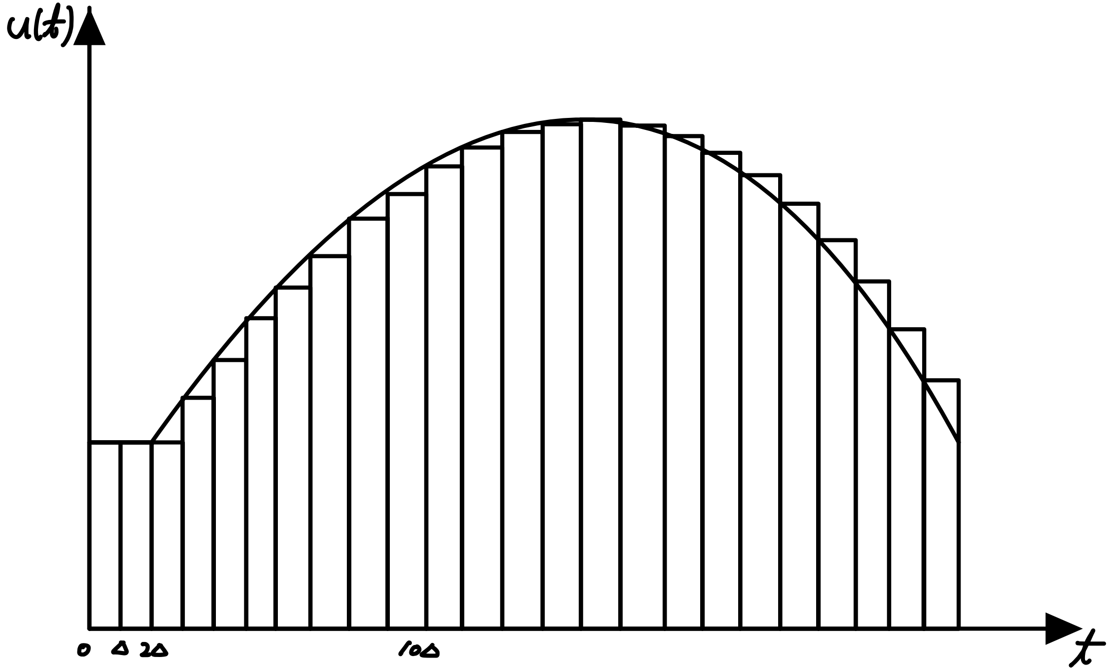
Convolution



- 입력이 $\sum_{k=0}^{\infty} u(k\Delta)\,\delta(t-k\Delta)$ 라면?

$$\sum_{k=0}^{\infty} \overset{\text{상수}}{u(k\Delta)}\, h(t-k\Delta)$$

Convolution



- 입력이 $\sum_{k=0}^{\infty} u(k\Delta)\, \delta(t - k\Delta)$ 라면?
- Input = $\int_0^{\infty} u(\tau)\delta(t-\tau)d\tau \rightarrow$ Output = $\int_0^{\infty} u(\tau)h(t-\tau)d\tau$

Superposition

$$y(t) = \int_0^{\infty} u(\tau)h(t-\tau)d\tau$$

- 어떤 **System**의 모든 $\overset{u(t)}{\textbf{input}}$에 대한 응답은 해당 system의 dirac delta function $\boldsymbol{\delta(t)}$에 대한 $\underset{h(t)}{\text{응답}}$만 알고 있다면 그 응답과 **convolution** 관계이다.

- 그럼, 매번 $\boldsymbol{\delta(t)}$에 대한 응답을 계산해서 convolution을 할 것인가?

- **Laplace!**

Laplace Transform

$(x^3)' = 3x^2 \to \frac{f(x+dh)-f(x)}{dh}$

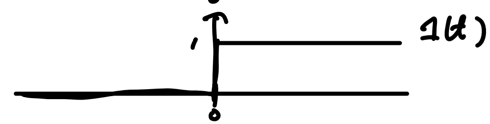
$$X(s) = \int_{0^-}^{\infty} e^{-st} x(t)dt$$

$$\begin{cases} 0 & (x > 0^+) \\ \infty & (0^- \le x \le 0^+) \\ 0 & (x < 0^-) \end{cases}$$

- Laplace Transform of Dirac Delta function $\delta(t)$

$$\delta(s) = \int_{0^-}^{\infty} e^{-st}\delta(t)\,dt = \int_{0^-}^{0^+} e^{-st}\delta(t)dt = \int_{0^-}^{0^+} e^{0}\delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

- Laplace Transform of unit step function 1(t)



$$1(s) = \int_{0^-}^{\infty} e^{-st}\,1(t)dt = \int_{0}^{\infty} e^{-st}dt = \left[-\frac{1}{s}e^{-st}\right]_0^{\infty} = \left(-\frac{1}{s}\times 0\right) - \left(-\frac{1}{s}\times 1\right) = \frac{1}{s}$$

- Laplace Transform of e^{at}, $t \ge 0$

$$F(s) = \int_{0^-}^{\infty} e^{-st}e^{at}dt = \int_{0}^{\infty} e^{-(s-a)t}dt = \left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^{\infty} = \frac{1}{s-a}$$

Laplace Transform의 성질

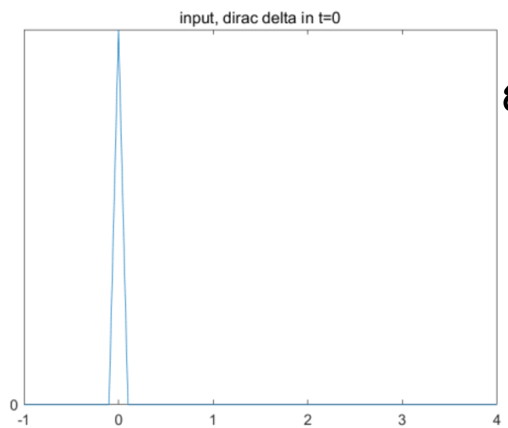
- 선형성
  - $\mathcal{L}(af(t) + bg(t)) = a\,\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ | $\mathcal{L}(e^{at}) \Rightarrow \frac{1}{s-a}$ | $\mathcal{L}\{e^{i\omega t}\} = \frac{1}{s-i\omega} = \frac{(s+i\omega)}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2}$

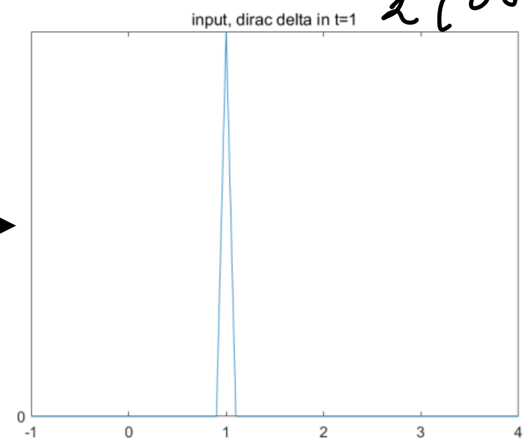
$\mathcal{L}(e^{i\omega t}) = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$ $\Rightarrow \mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}$, $\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$

- t-shifting
  - $\mathcal{L}\{f(t-a)1(t-a)\} = e^{-as}F(s)$

$\mathcal{L}\{\delta(t-1)\} = e^{-s}$



$\mathcal{L}\{\delta(t)\} = 1$



Laplace Transform의 성질

- t-scaling
  - $\mathcal{L}(f(at)) = \frac{1}{a} F(\frac{s}{a})$ $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$ $\Rightarrow$ $\mathcal{L}\{\sin(2\omega t)\} = \frac{1}{2} \times \frac{\omega}{(\frac{s}{2})^2+\omega^2}$

- s-shifting
  - $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
  - $\mathcal{L}\{e^{at} \cos \omega t\}$

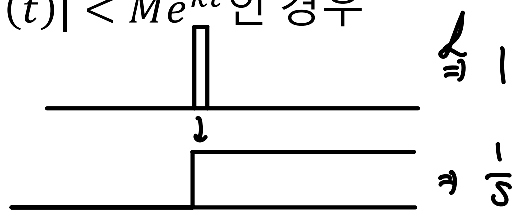
    $\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$ $\Rightarrow$ $\mathcal{L}(e^{-t} \cos \omega t) = \frac{(s+1)}{(s+1)^2+\omega^2}$
  - $\mathcal{L}\{e^{at} \sin \omega t\}$

$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$ $\Rightarrow$ $\mathcal{L}(e^{-2t} \sin \omega t) = \frac{\omega}{(s+2)^2+\omega^2}$

Laplace Transform의 성질

- Time-domain 미분
  - $\mathcal{L}(f'(t)) = s\mathcal{L}\{f\} - f(0)$
  - $\mathcal{L}(f''(t)) = s\mathcal{L}\{f'\} - f'(0) = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$

- Time-domain 적분
  - $f(t)$ $piecewise$ 연속, $|f(t)| < Me^{kt}$인 경우
  - $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s)$



Laplace Transform의 성질

- Time product
  - $\mathcal{L}\big(tf(t)\big) = -\frac{d}{ds}F(s)$
  - $\mathcal{L}(t \sin \omega t)$

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}(t\sin\omega t) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = -\left\{-\frac{\omega\cdot(s^2+\omega^2)'}{(s^2+\omega^2)^2}\right\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Laplace Transform의 성질

- Convolution
  - $$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s)$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s-a}\right\}$

$*\mathcal{L}\{1(t)\} = \frac{1}{s}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$\int_0^t \tau e^{a(t-\tau)}d\tau = e^{at}\int_0^t \tau e^{-a\tau}d\tau = \frac{1}{a^2}\left(e^{at}-at-1\right)$$

Solving ODEs by Laplace transform

- System : $\dot{y} + ky = u \;\leftarrow$ **LTI**
- Input $u = \delta(t)$

$f(x) = f(0) + f'(0)x + \sim$ $\quad \boxed{f(x) - f(0)} = \sim$

변동량

$$\dot{y}(t) + k\,y(t) = \delta(t)$$

$$\int_{0^-}^{0^+} \dot{y}\,dt + k\int_{0^-}^{0^+} \cancel{y}\,dt = 1 \qquad (\to 0)$$

$$\int_{0^-}^{0^+} \dot{y}\,dt = 1 \;\Rightarrow\; \underline{y(0^+) = 1}, \quad \dot{y} + ky = 0$$

$$y = Ae^{\lambda t} \Rightarrow \lambda = -k \Rightarrow A = 1$$

$$y(t) = e^{-kt}$$

$$\dot{y}(t) + k\,y(t) = \delta(t)$$

(제어공학 I.C. ⇒ all 0)

$$sY(s) + \cancel{y(0)} + kY(s) = 1 \qquad (\to 0)$$

$$(s+k)\,Y(s) = 1 \longrightarrow Y(s) = \frac{1}{s+k}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = e^{-kt}$$