Chapter 2
Vector Analysis

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Electromagnetic Equations

- Maxwell's Equations

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \bullet \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu\left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

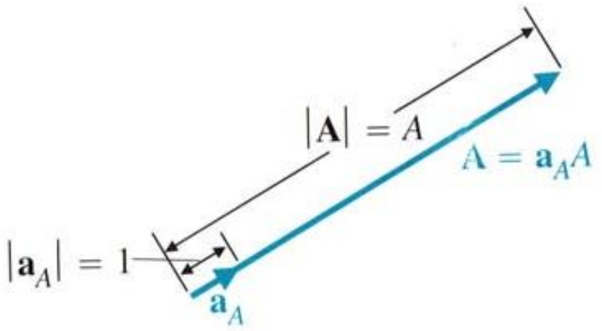
Chapter 2. Vector Analysis

- Physical quantities
  - Scalar – magnitude
  - Vector – magnitude and direction
  - Tensor

- The laws of electromagnetism do not require the specification of a coordinate system.
  - We can choose a particular coordinate system
  - Rectangular coordinates if a current loop is rectangular
  - Polar coordinates if the current loop is circular
- Coordinate system
  - Cartesian
  - Cylindrical
  - Spherical

Vector Analysis

- Vector



- Vector addition and Subtraction

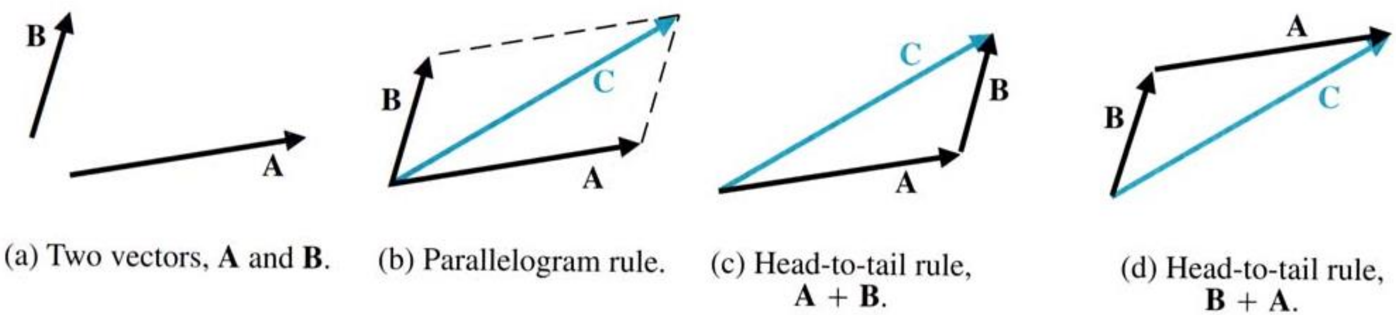


FIGURE 2-2 Vector addition, **C** = **A** + **B** = **B** + **A**.

- Vector multiplication
  - $k\mathbf{A}=\mathbf{a_A}\,(kA)$

Vector Analysis

- Scalar product $\mathbf{A}\bullet\mathbf{B} = AB\cos\theta_{AB}$

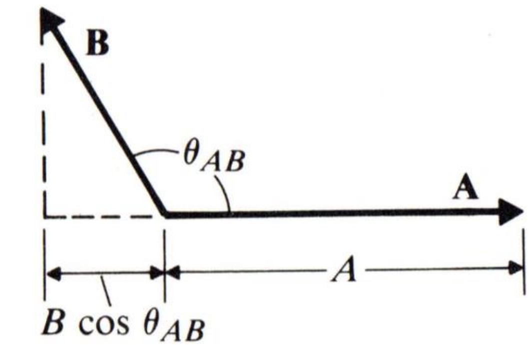
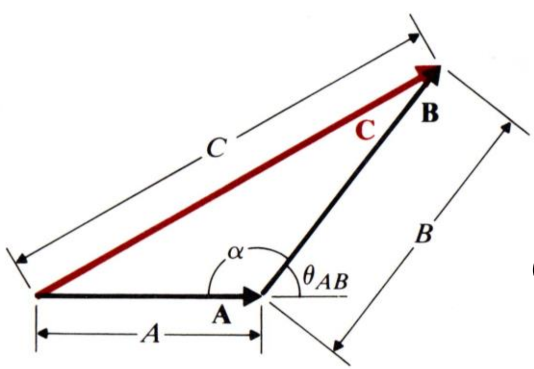


FIGURE 2–4
Illustrating the dot product of **A** and **B**.

- Cosines for a triangle (ex 2-1)

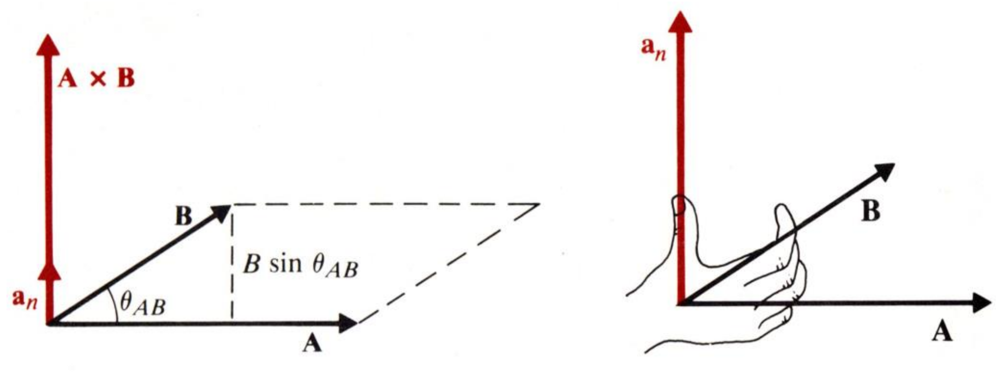


$$C^2 = \mathbf{C}\bullet\mathbf{C} = (\mathbf{A}+\mathbf{B})\bullet(\mathbf{A}+\mathbf{B}) = A^2 + B^2 + 2AB\cos\theta_{AB}$$

Vector Analysis

- Vector product

$$\mathbf{A}\times\mathbf{B}=\mathbf{a}_n\left|AB\sin\theta_{AB}\right|$$



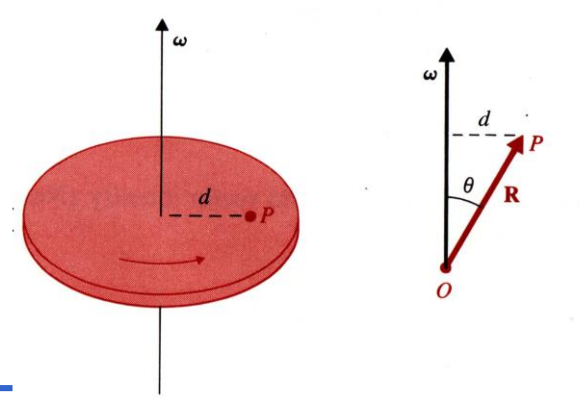
(a) $\mathbf{A}\times\mathbf{B}=\mathbf{a}_n|AB\sin\theta_{AB}|$. (b) The right-hand rule.

- Triple product (not associative)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\neq(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

- Ex 2-2

$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{R}$$



(a) A rotating disk. (b) Vector representation.

Vector Analysis

- Products of Three Vectors
  - **A**·(**B**×**C**) = **B**·(**C**×**A**) = **C**·(**A**×**B**)

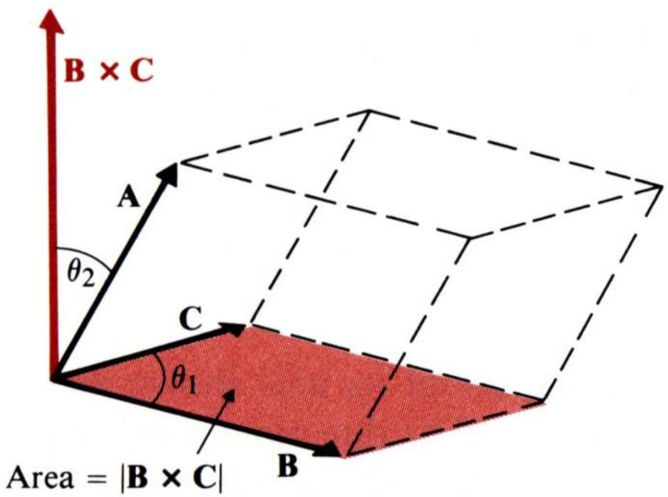


FIGURE 2–8
Illustrating scalar triple product **A** · (**B** × **C**).

  - **Back-cab rule**
    - **A**×(**B**×**C**) = ? ➔ **B**(**A**·**C**)-**C**(**A**·**B**)
    - Proof in example 2-3

Orthogonal coordinate systems

- Cartesian, cylindrical, spherical coordinates
- In 3D space, the three families of surface are described by u_1=const, u_2=const and u_3 =const
- In Cartesian coordinate system
  - $u_1 = x$, $u_2 = y$ and $u_3 = z$

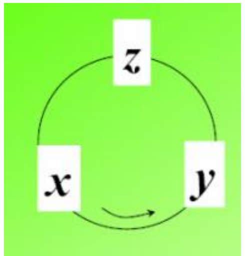
$$\mathbf{a_x} \times \mathbf{a_y} = \mathbf{a_z}, \mathbf{a_y} \times \mathbf{a_z} = \mathbf{a_x}, \mathbf{a_z} \times \mathbf{a_x} = \mathbf{a_y}$$

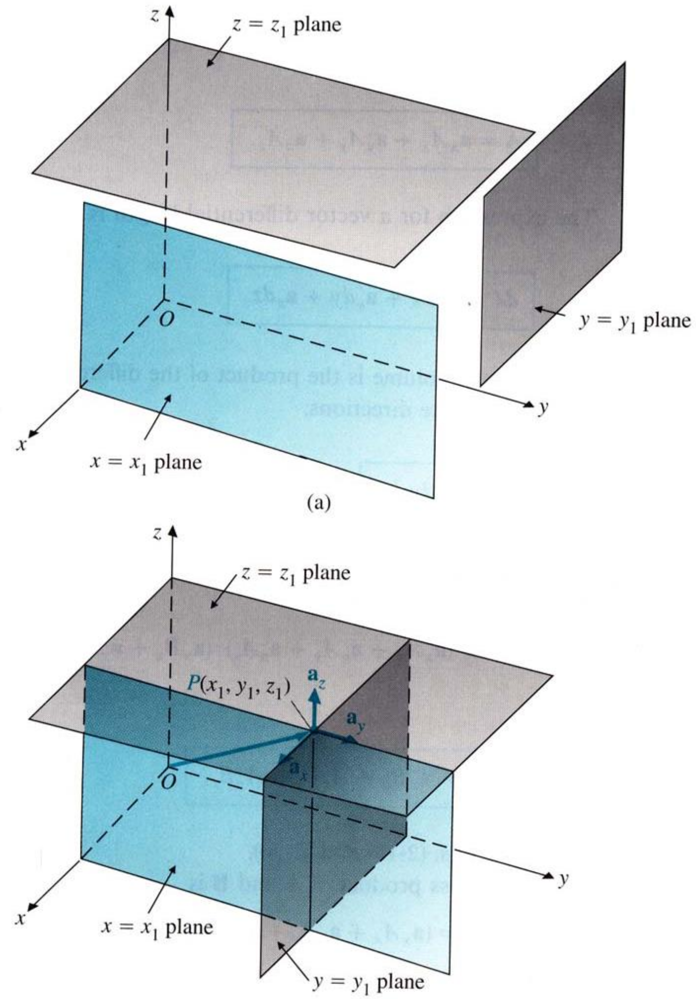
$$\mathbf{a_x} \bullet \mathbf{a_y} = \mathbf{a}_y \bullet \mathbf{a}_z = \mathbf{a}_x \bullet \mathbf{a}_z = 0$$

$$\mathbf{OP} = \mathbf{a_x} x_1 + \mathbf{a_y} y_1 + \mathbf{a_z} z_1$$

$$d\mathbf{l} = \mathbf{a_x} dx + \mathbf{a_y} dy + \mathbf{a_z} dz$$

$$d\upsilon = dxdydz$$





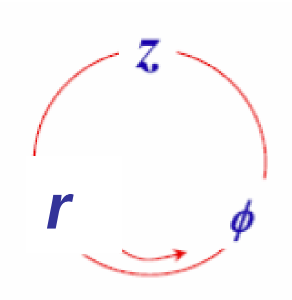
Cylindrical coordinates

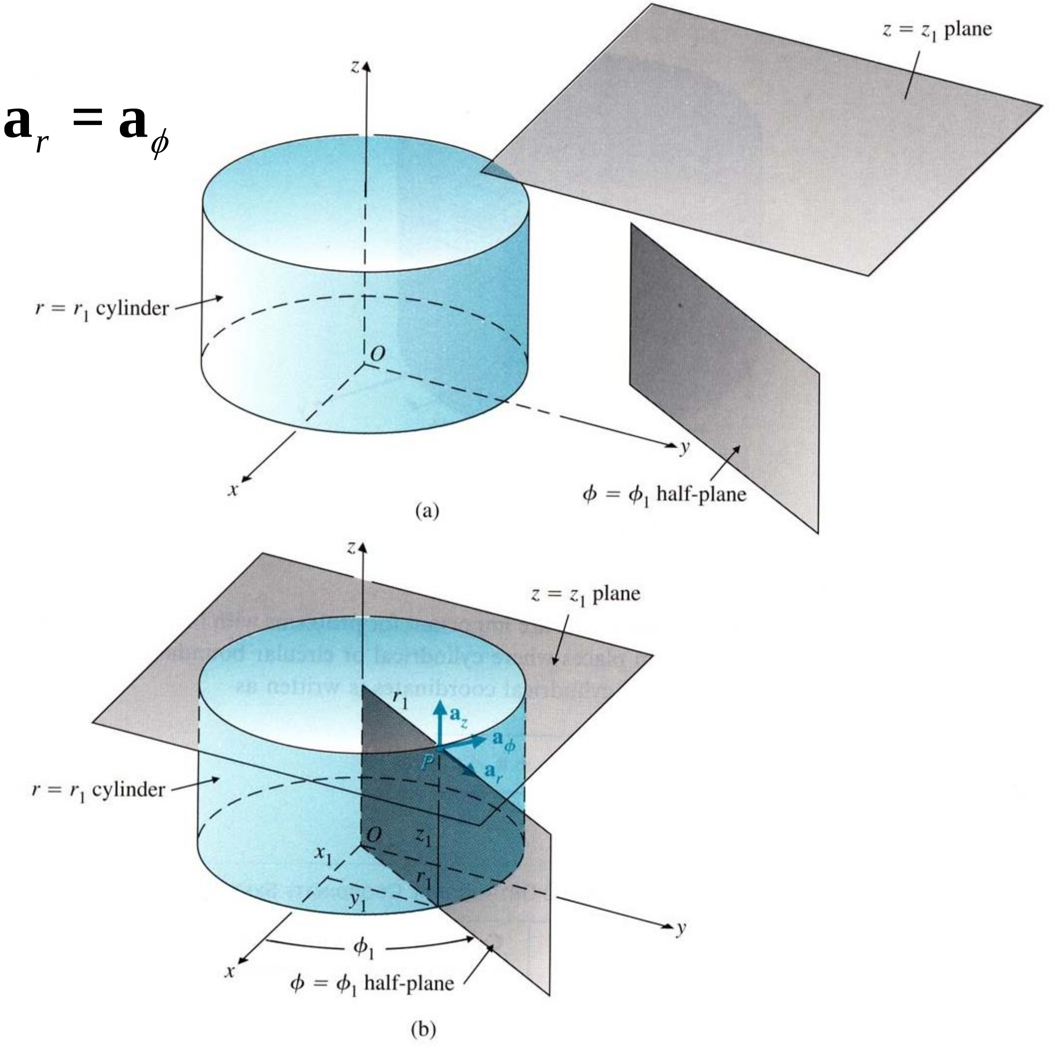
- $u_1, u_2, u_3 = (r, \phi, z)$

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z, \mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r, \mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi$$

$$dl = \mathbf{a}_r dr + \mathbf{a}_\phi r d\phi + \mathbf{a}_\mathbf{z} dz$$

$$d\upsilon = r dr d\phi dz$$

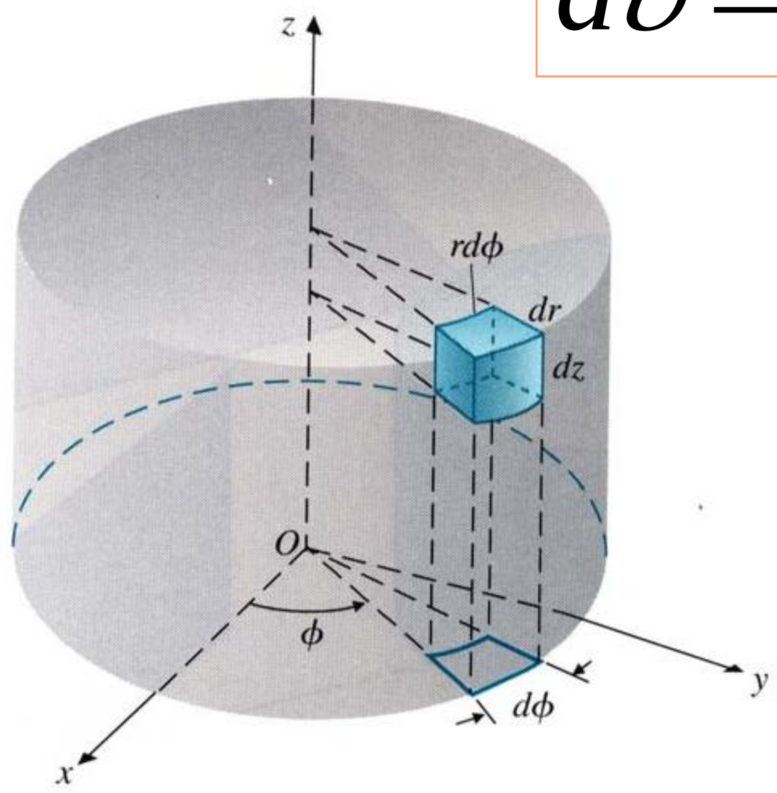




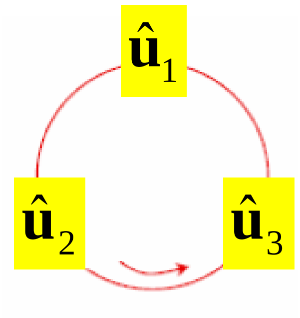
- Example 2.9

Cylindrical coordinates

- Differential volume element



$$d\upsilon = r dr d\phi dz$$



Cylindrical coordinates

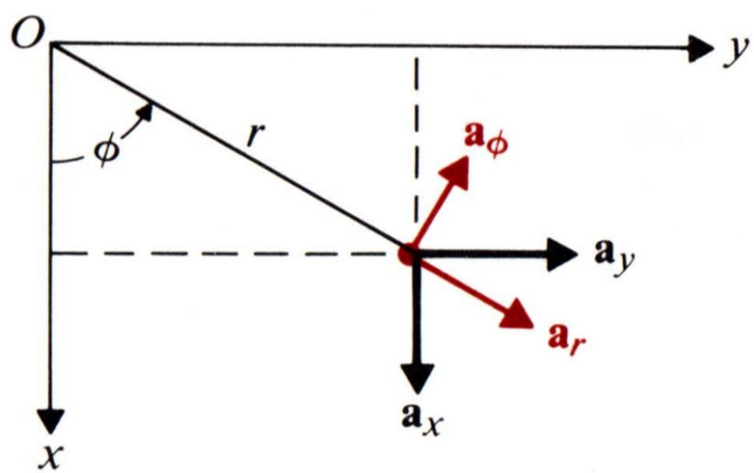
- Vector transformation

$$\mathbf{A} = A_r\mathbf{a}_r + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z \rightarrow A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

$$\mathbf{a}_r \bullet \mathbf{a}_x = \cos\phi$$

$$\mathbf{a}_\phi \bullet \mathbf{a}_x = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$$

$$\mathbf{a}_r \bullet \mathbf{a}_y = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi$$



$$A_x = A_r\cos\phi - A_\phi\sin\phi$$

$$A_y = A_r\sin\phi + A_\phi\cos\phi$$

$$A_z = A_z$$

Spherical Coordinates

- $u_1, u_2, u_3 = (R, \theta, \phi)$

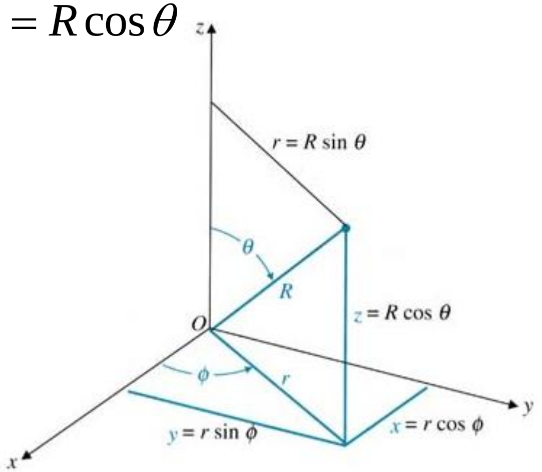
$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi, \mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R, \mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta$$

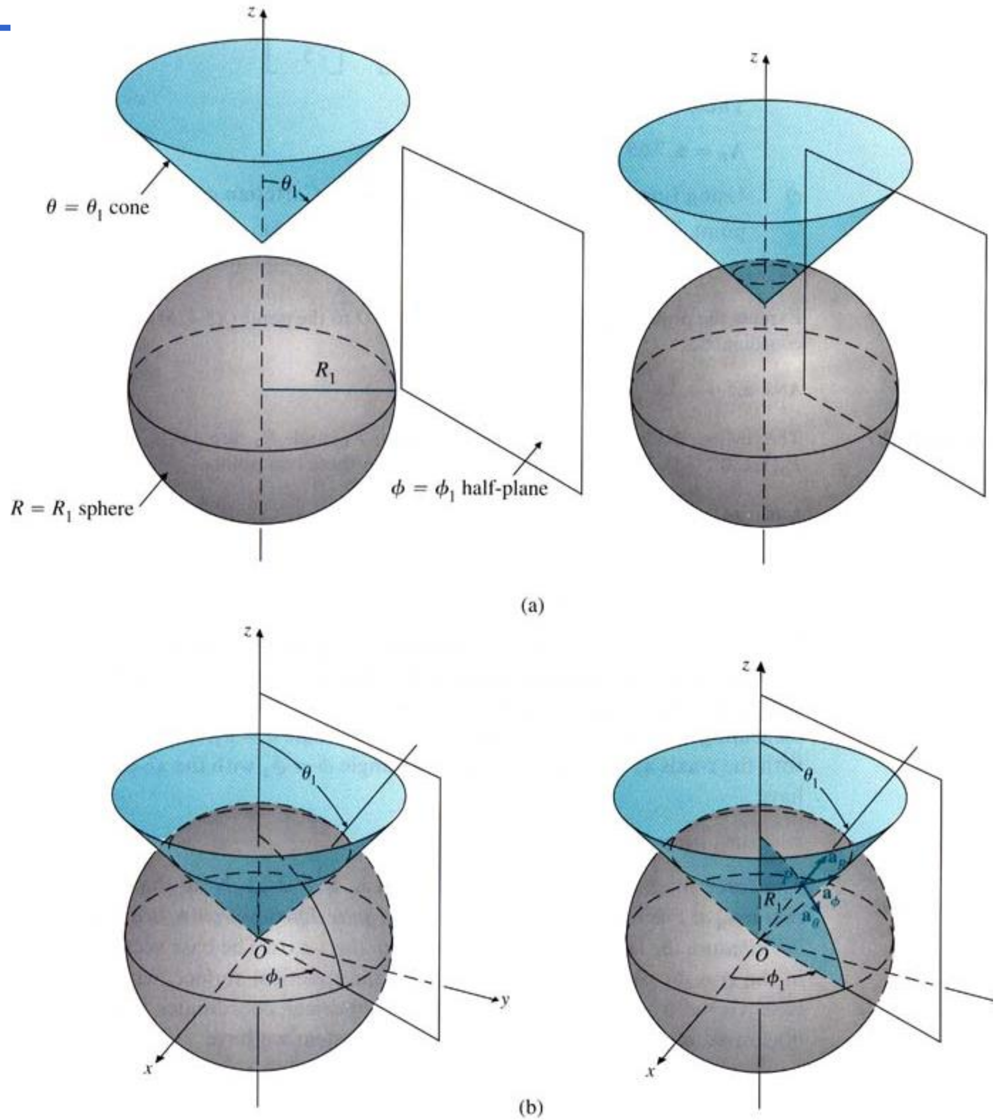
$$d\mathbf{l} = dR\ \mathbf{a}_R + Rd\theta\ \mathbf{a}_\theta + R\sin\theta d\phi\ \mathbf{a}_\phi$$

$$d\upsilon = R^2 \sin\theta dR d\theta d\phi$$

- Example 2.10, 11,12

$$x = R\sin\theta\cos\phi, y = R\sin\theta\sin\phi,$$
$$z = R\cos\theta$$

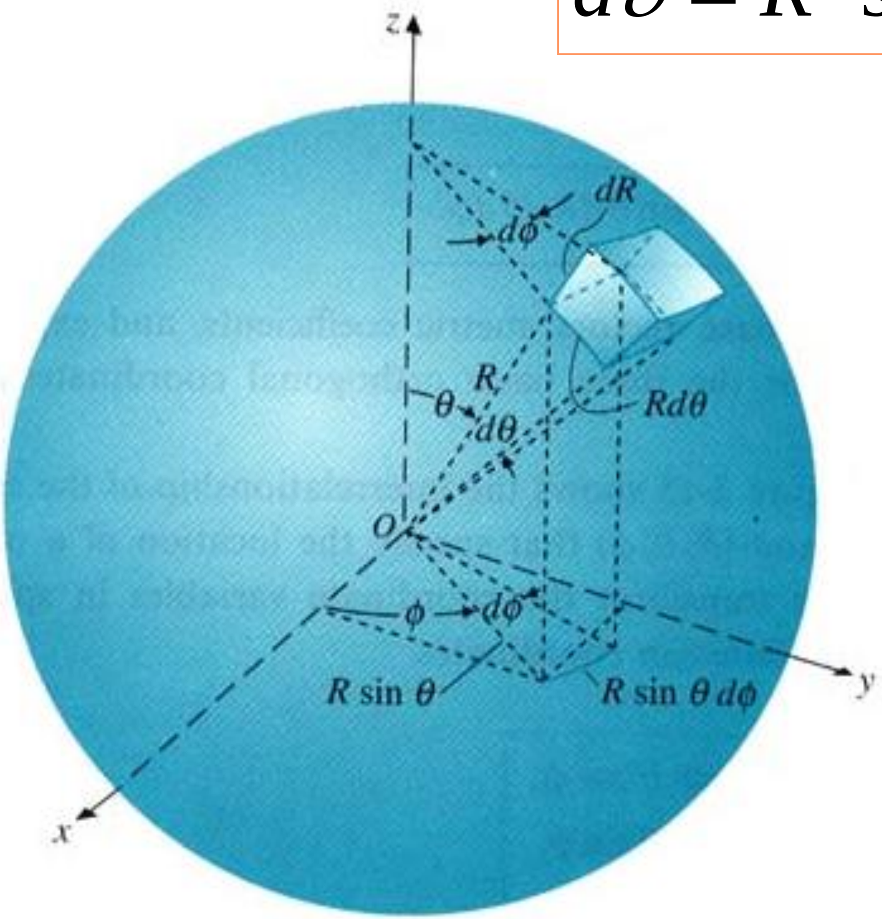




Spherical Coordinates

- Differential volume

$$d\upsilon = R^2 \sin\theta dR d\theta d\phi$$



Metric Coefficients

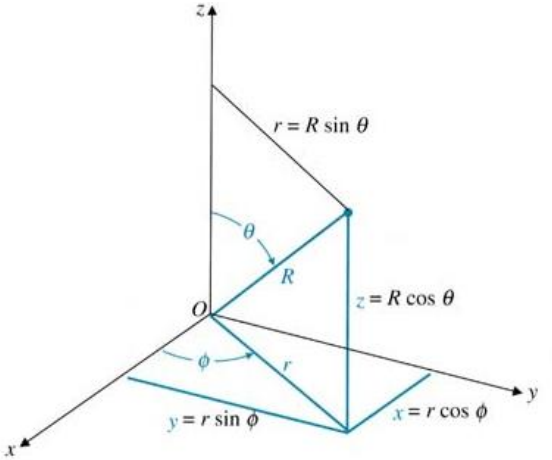
	h_1	h_2	h_3
x, y, z	1	1	1
r, ϕ, z	1	r	1
R, θ, ϕ	1	R	$R \sin\theta$

$$d\upsilon = h_1 h_2 h_3 du_1 du_2 du_3$$

Spherical Coordinates

- Example 2.11
  - Converting a vector into Cartesian coordinates

$$\mathbf{A} = A_R \mathbf{a}_R + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi$$



$$A_x = A_R \mathbf{a}_R \bullet \mathbf{a}_x + A_\theta \mathbf{a}_\theta \bullet \mathbf{a}_x + A_\phi \mathbf{a}_\phi \bullet \mathbf{a}_x$$

$$A_x = A_R \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$$

$$A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$$

$$A_z = A_R \cos\theta - A_\theta \sin\theta$$

Generalized Orthogonal Coordinate

- Base vectors $\mathbf{a}_{u_1}, \mathbf{a}_{u_2}, \mathbf{a}_{u_3}$

$$\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}$$

- Displacement vector

$$d\mathbf{l} = \mathbf{a}_{u_1}(h_1 du_1) + \mathbf{a}_{u_2}(h_2 du_2) + \mathbf{a}_{u_3}(h_3 du_3)$$

$$dl = \sqrt{(h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^3}$$

- Differential volume $dv = h_1 h_2 h_3 du_1 du_2 du_3$

- Differential area $d\mathbf{s} = \mathbf{a}_n ds,$

$$ds_1 = h_2 h_3 du_2 du_3, \quad ds_2 = h_1 h_3 du_1 du_3$$

Generalized Orthogonal Coordinate

Coordinate System Relations	Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, Φ, z)	Spherical Coordinates (R, θ, Φ)
Base vectors $\mathbf{a}_{u_1}$ $\mathbf{a}_{u_2}$ $\mathbf{a}_{u_3}$	a_x a_y a_z	a_r a_Φ a_z	a_R a_θ a_Φ
Metric coefficients h_1 h_2 h_3	1 1 1	1 r 1	1 R R sin θ
Differential volume dv	dx dy dz	r dr dΦ dz	R^2 sin θ dR dθ dΦ

Integrals containing vector functions

- Integrals

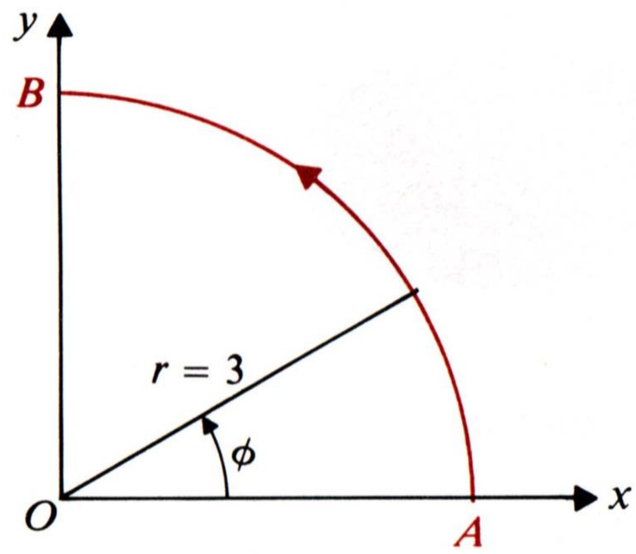
$$\int_V \mathbf{F}\,dv, \quad \int_C V\,d\mathbf{l}$$

$$\int_C \mathbf{F}\bullet d\mathbf{l}, \quad \int_S \mathbf{A}\bullet d\mathbf{s}$$

- Example 2.13,14,15

Integrals containing vector functions

- Example 2-14



$$\mathbf{F} = \mathbf{a}_x xy - \mathbf{a}_y 2x$$

$$\int_A^B \mathbf{F} \cdot d\mathbf{l} = ?$$

  – In Cartesian coordinates

$$\int_A^B \mathbf{F} \cdot d\mathbf{l} = \int_3^0 x\sqrt{9-x^2}\,dx - 2\int_0^3 \sqrt{9-y^2}\,dy = -9\left(1+\frac{\pi}{2}\right)$$

  – In cylindrical coordinates

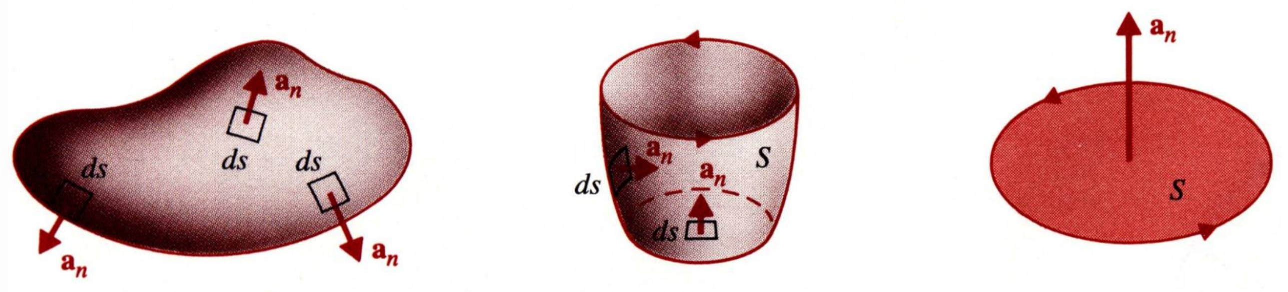
$$\mathbf{F} = \mathbf{a}_r\left(xy\cos\phi - 2x\sin\phi\right) - \mathbf{a}_\phi\left(xy\sin\phi + 2x\cos\phi\right)$$

$$d\mathbf{l} = \mathbf{a}_\phi 3d\phi$$

Integrals containing vector functions

- Surface integral
$$\int_S \mathbf{A} \bullet d\mathbf{s}$$
  - A closed surface

$$\oint_S \mathbf{A} \bullet d\mathbf{s} = \oint_S \mathbf{A} \bullet \mathbf{a}_n \, ds$$



(a) A closed surface. (b) An open surface. (c) A disk.

FIGURE 2–22
Illustrating the positive direction of $\mathbf{a}_n$ in scalar surface integral.

Integrals containing vector functions

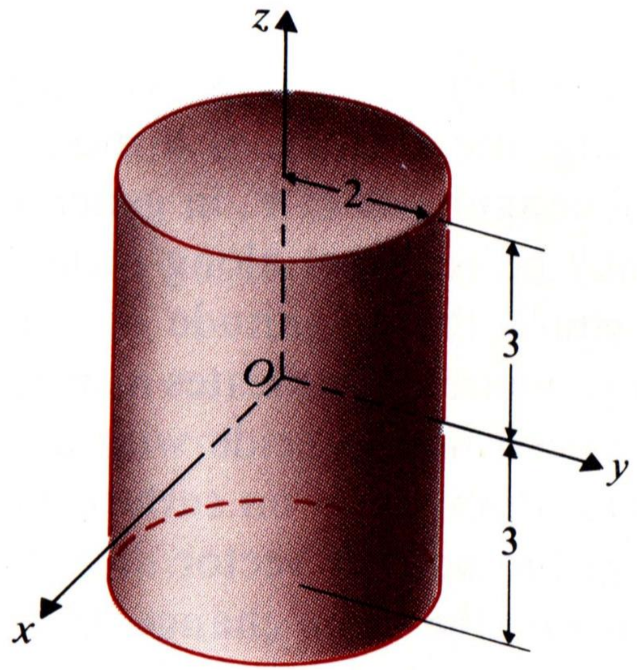
- Example 2-15

$$\mathbf{F} = \mathbf{a}_r \frac{k_1}{r} + \mathbf{a}_z k_2 z$$

$$\oint \mathbf{F} \cdot d\mathbf{s} = \int_{\text{top surface}} \mathbf{F} \cdot \mathbf{a}_n ds + \int_{\text{bottom surface}} \mathbf{F} \cdot \mathbf{a}_n ds + \int_{\text{side wall}} \mathbf{F} \cdot \mathbf{a}_n ds$$

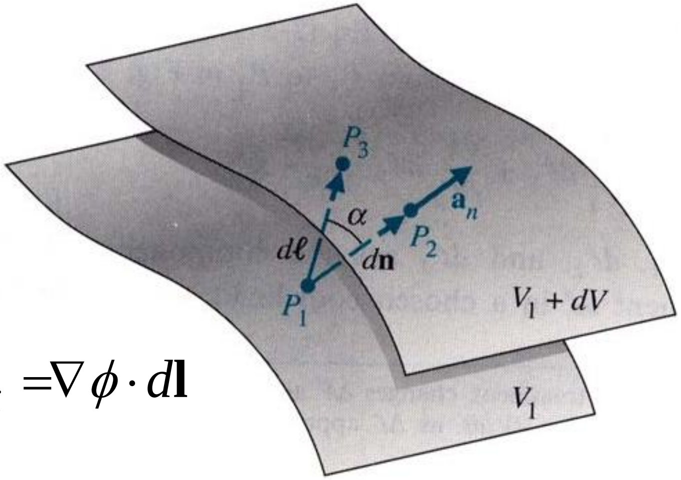
$$= 12\pi \left(k_1 + 2k_2 \right)$$

  – Outward flux of the vector **F**



Gradient of a Scalar Field

- Gradient : the vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar.



$$d\phi = \sum_i \frac{\partial \phi}{\partial u_i} du_i = \sum_i \frac{\partial \phi}{h_i \partial u_i} h_i du_i = \sum_i \left(\nabla \phi\right)_i \cdot dl_i = \nabla \phi \cdot d\mathbf{l}$$

$$\leftarrow d\mathbf{l} = (h_1 du_1, h_2 du_2, h_3 du_3), \nabla \phi = \sum_i \mathbf{a}_{u_i} \frac{\partial \phi}{h_i \partial u_i}$$

Maximum value of $d\phi$

$$\left(d\phi\right)_{\max} = \left|\nabla \phi\right| dl, \text{ for } \nabla \phi \parallel d\mathbf{l}$$

$$d\phi = 0, \text{ for } \nabla \phi \perp d\mathbf{l}$$

Gradient of a Scalar Field

- Example 2.16

$$(u_1, u_2, u_3) = (x, y, z)$$

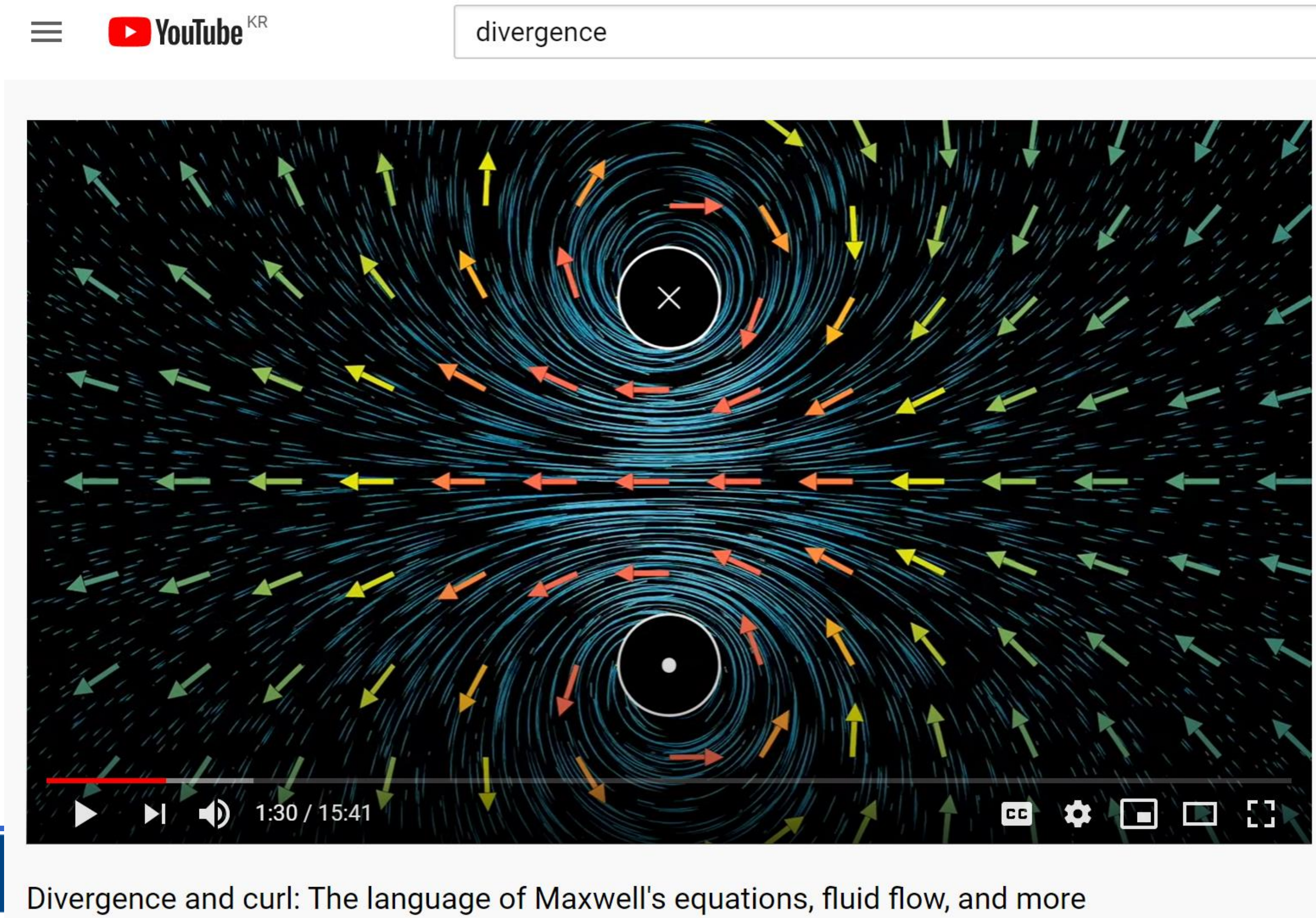
$$\nabla V = \left(\mathbf{a_x} \frac{\partial}{\partial x} + \mathbf{a_y} \frac{\partial}{\partial y} + \mathbf{a_z} \frac{\partial}{\partial z} \right) V$$

In general orthogonal coordinates (u_1, u_2, u_3)

$$\nabla \equiv \left(\mathbf{a}_{u_1} \frac{\partial}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3 \partial u_3} \right)$$

Vector Field

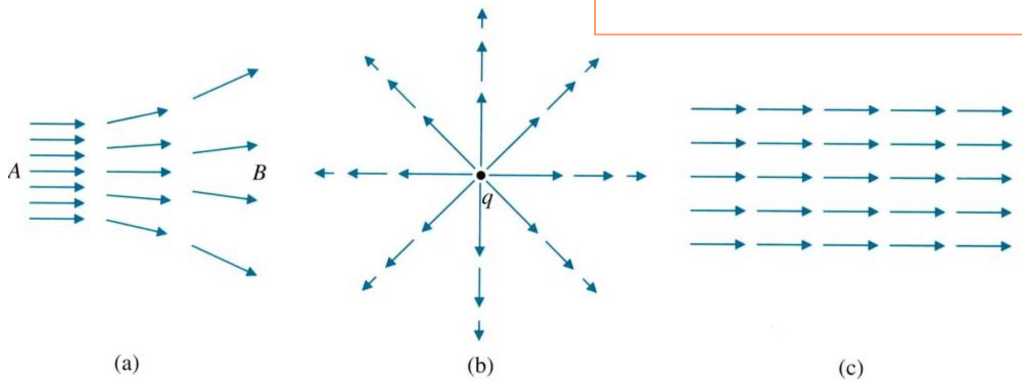
- Divergence & curl
  - 3Blue1Brown
  - https://www.youtube.com/watch?v=rB83DpBJQsE&t=1026s



Divergence of a Vector Field

- Flux lines : representation of field variations graphically by directed field lines.
- Magnitude of the field at a point : either depicted by the density or by the length of the directed lines in the vicinity of the point
- Divergence at a point: the net outward flux of **A** per unit volume as the volume about the point tends to zero

$$div\ \mathbf{A} = \nabla \bullet \mathbf{A} \triangleq \lim_{\Delta v \to 0} \frac{\oint_s \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

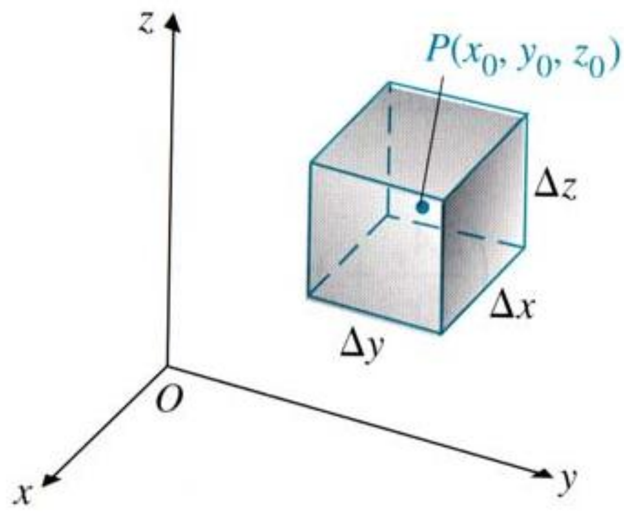


The flow of an incompressible fluid

Divergence of a Vector Field

- $$\nabla \bullet \mathbf{A} \triangleq \lim_{\Delta \upsilon \to 0} \frac{\oint_s \mathbf{A} \cdot d\mathbf{s}}{\Delta \upsilon}$$

$$\to \oint_s \mathbf{A} \cdot d\mathbf{s} = \left[\int_{front} + \int_{back} + \int_{right} + \int_{left} + \int_{top} + \int_{bottom} \right] \mathbf{A} \cdot d\mathbf{s}$$



Divergence of a Vector Field

- On the front face

$$\int_{front} \mathbf{A}\cdot d\mathbf{s} = \mathbf{A}_{front}\cdot\Delta\mathbf{s}_{front}$$

$$= \mathbf{A}_{front}\cdot\mathbf{a}_x\Delta y\Delta z = A_x(x_0+\frac{\Delta x}{2}, y_0, z_0)\Delta y\Delta z$$

$$A_x(x_0+\frac{\Delta x}{2}, y_0, z_0) \approx A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + \text{higher-order terms}$$

$$\int_{front} \mathbf{A}\cdot d\mathbf{s} = \Delta y\Delta z\left(A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + \text{higher-order terms}\right)$$

Divergence of a Vector Field

- On the front face

$$*\int_{back} \mathbf{A}\cdot d\mathbf{s} = \mathbf{A}_{back}\cdot\Delta\mathbf{s}_{back} = \mathbf{A}_{back}\cdot -\mathbf{a}_x\left(\Delta y\Delta z\right) = -A_x(x_0-\frac{\Delta x}{2}, y_0, z_0)\Delta y\Delta z$$

$$A_x(x_0-\frac{\Delta x}{2}, y_0, z_0) = A_x(x_0, y_0, z_0) - \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + \text{higher-order terms}$$

$$\left[\int_{front} + \int_{back}\right]\mathbf{A}\cdot d\mathbf{s} \approx \left(\frac{\partial A_x}{\partial x}\right)_{(x_0,y_0,z_0)}\Delta x\Delta y\Delta z$$

Divergence of a Vector Field

- Following the same procedure for 4 faces

$$\oint_s \mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)_{(x,y,z)} \Delta x \Delta y \Delta z$$

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence of a Vector Field

- In general orthogonal curvilinear coordinates (u_1,u_2,u_3)

$$\nabla \bullet \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(h_2 h_3 A_1\right) + \frac{\partial}{\partial u_2}\left(h_1 h_3 A_2\right) + \frac{\partial}{\partial u_3}\left(h_1 h_2 A_3\right)\right]$$

- Find the divergence of the position vector to an arbitrary point
  - Example 2.17

$$\mathbf{OP} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z \rightarrow \nabla \bullet \mathbf{OP} = 3$$

$$\mathbf{OP} = \mathbf{a}_R R \rightarrow \nabla \bullet \mathbf{OP} = 3$$

- Find the divergence of the magnetic flux density B outside a very long current-carrying wire
  - Example 2.18

$$\mathbf{B} = \mathbf{a}_\phi \frac{k}{r} \rightarrow \nabla \bullet \mathbf{B} = 0$$

Divergence Theorem

- $$\int_V \nabla\cdot\mathbf{A}\,d\upsilon = \oint_S \mathbf{A}\cdot d\mathbf{s}$$

For a very small differential volume element $\Delta\upsilon_j$ bounded by a surface s_j

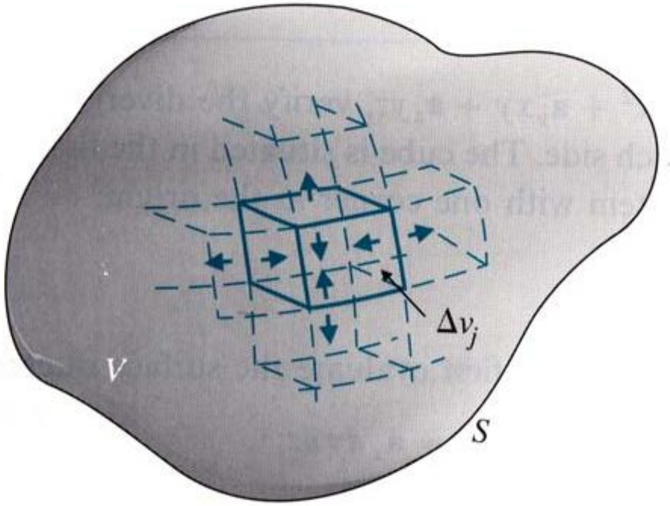
$$(\nabla\cdot\mathbf{A})_j\,\Delta\upsilon_j = \oint_{s_j}\mathbf{A}\cdot d\mathbf{s} \leftarrow (\nabla\cdot\mathbf{A})_j = \frac{\oint_{s_j}\mathbf{A}\cdot d\mathbf{s}}{\Delta\upsilon_j}$$

$$\lim_{\Delta\upsilon\to 0}\left[\sum_{j=1}^{N}(\nabla\cdot\mathbf{A})_j\,\Delta\upsilon_j\right] = \lim_{\Delta\upsilon\to 0}\left[\sum_{j=1}^{N}\oint_{s_j}\mathbf{A}\cdot d\mathbf{s}\right]$$

$$\lim_{\Delta\upsilon\to 0}\left[\sum_{j=1}^{N}(\nabla\cdot\mathbf{A})_j\,\Delta\upsilon_j\right] = \int_V \nabla\cdot\mathbf{A}\,d\upsilon$$

$$\lim_{\Delta\upsilon\to 0}\left[\sum_{j=1}^{N}\oint_{s_j}\mathbf{A}\cdot d\mathbf{s}\right] = \oint_S \mathbf{A}\cdot d\mathbf{s}$$

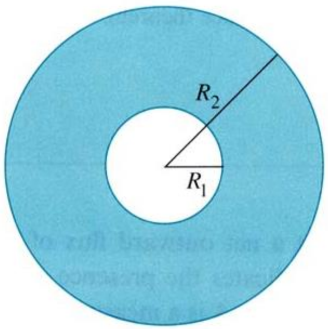
$$\therefore\ \int_V \nabla\cdot\mathbf{A}\,d\upsilon = \oint_S \mathbf{A}\cdot d\mathbf{s}$$



- Requirement : **A** and its first derivatives exists and be continuous both in V and on S
- Example 2-19

Divergence Theorem

- Example 2-20
  - **F** = **a**$_R$ *kR*, determine whether the divergence theorem holds for the shell region



at the outer surface: $R = R_2, d\mathbf{s} = \mathbf{a}_R R_2^2 \sin\theta d\theta d\phi$

$$\int_{\substack{\text{outer}\\\text{surface}}} \mathbf{F}\bullet d\mathbf{s} = \int_0^{2\pi}\int_0^{\pi}(kR_2)R_2^2\sin\theta d\theta d\phi = 4\pi k R_2^3$$

at the inner surface: $R = R_1, d\mathbf{s} = -\mathbf{a}_R R_1^2 \sin\theta d\theta d\phi$

$$\int_{\substack{\text{inner}\\\text{surface}}} \mathbf{F}\bullet d\mathbf{s} = \int_0^{2\pi}\int_0^{\pi}(kR_1)R_1^2\sin\theta d\theta d\phi = 4\pi k R_1^3$$

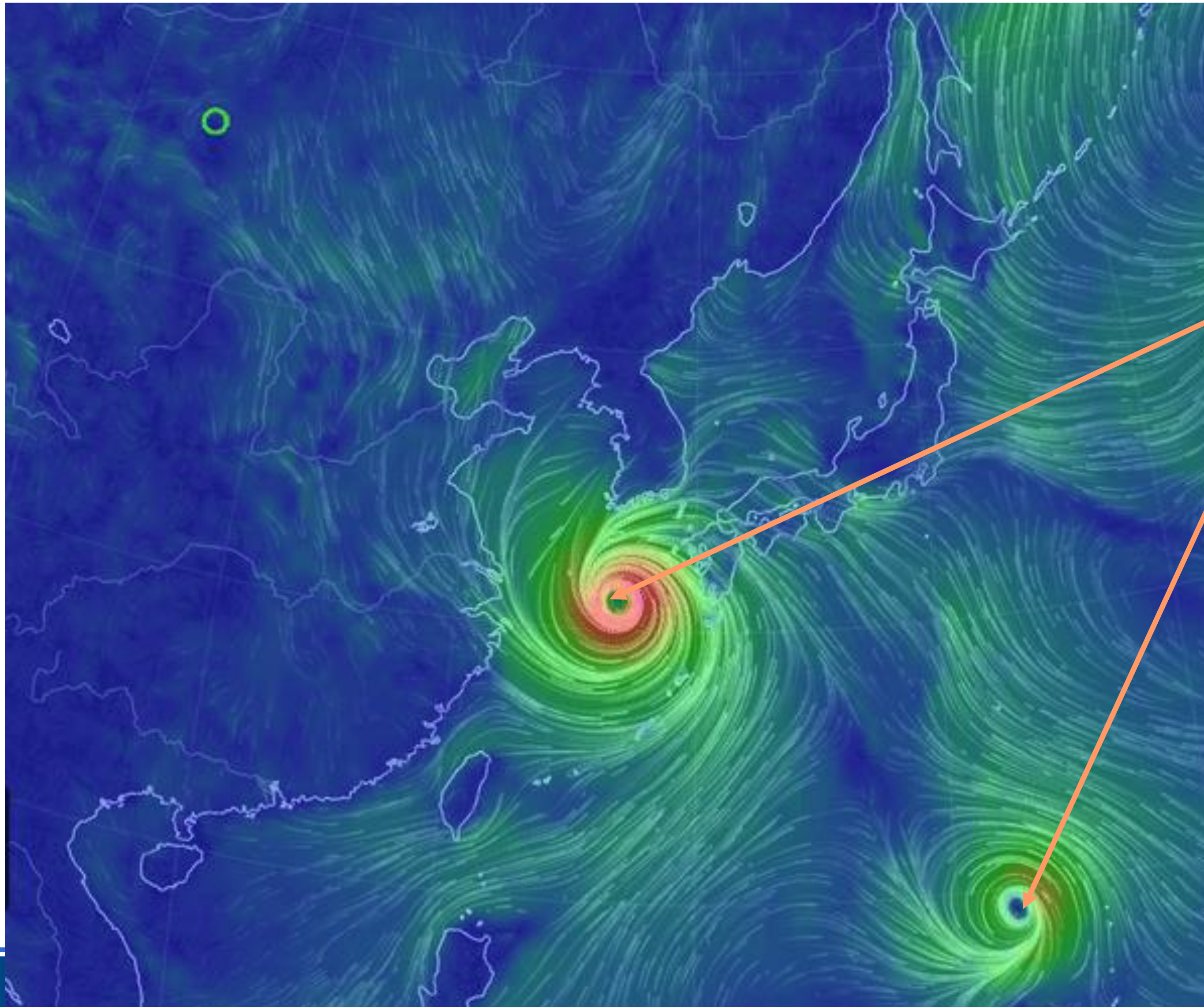
$$\oint_S \mathbf{F}\bullet d\mathbf{s} = 4\pi k\left(R_2^3 - R_1^3\right)$$

$$\nabla\bullet\mathbf{F} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 F_R\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(kR^3\right) = 3k$$

$$\int_V \nabla\bullet\mathbf{F}d\upsilon = (\nabla\bullet\mathbf{F})V = 3k\frac{4\pi\left(R_2^3 - R_1^3\right)}{3} = 4\pi k\left(R_2^3 - R_1^3\right)$$

Curl of A Vector Field

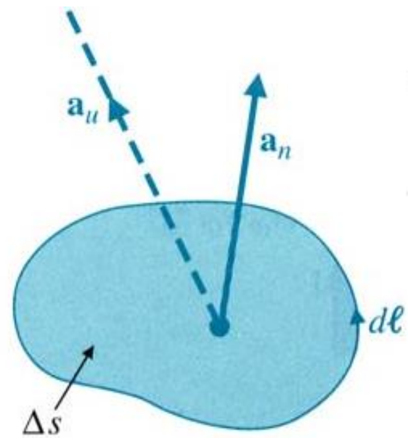
- Air flow : https://earth.nullschool.net



Curl of A Vector Field

- Circulation of **A** around contour C

  - $$\oint_C \mathbf{A} \cdot dl$$



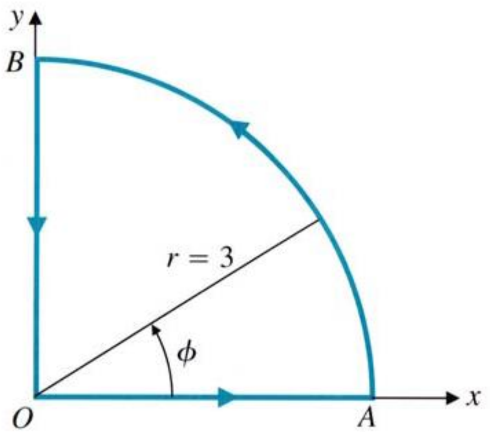
- Curl A

$$\nabla \times \mathbf{A} \triangleq \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\mathbf{a}_n \oint_C \mathbf{A} \cdot dl \right]$$

- A vector whose magnitude is the maximum net circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum.

Curl of A Vector Field

- Example 2-22
  - Given a vector field $\mathbf{F} = \mathbf{a}_x xy - \mathbf{a}_y 2x$
  - Find its circulation around the path *OABO*



$$\oint_{OABO} \mathbf{F} \bullet dl = \int_O^A \mathbf{F} \bullet dl + \int_A^B \mathbf{F} \bullet dl + \int_B^O \mathbf{F} \bullet dl$$

along path *OA*: $y = 0, \mathbf{F} = -\mathbf{a}_y 2x, dl = \mathbf{a}_x dx, \mathbf{F} \bullet dl = 0$

$$\int_O^A \mathbf{F} \bullet dl = 0$$

along path *BO*: $x = 0, \mathbf{F} = \mathbf{0}, \int_B^O \mathbf{F} \bullet dl = 0.$

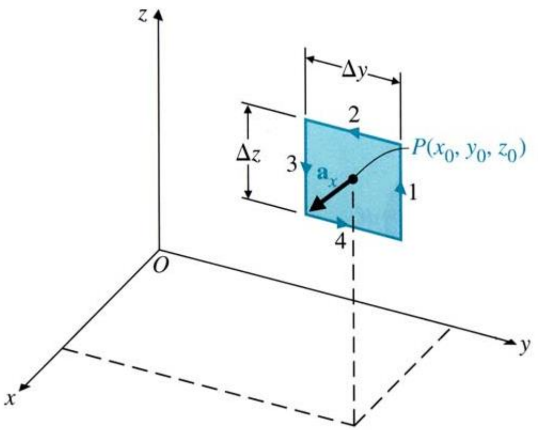
along path *AB*: $dl = \mathbf{a}_x dx + \mathbf{a}_y dy, \mathbf{F} \bullet dl = xydx - 2xdy$

The equation of the quarter-circle is $x^2 + y^2 = 9 \left(0 \le x \le 3\right)$

$$\int_A^B \mathbf{F} \bullet dl = \int_3^0 x\sqrt{9 - x^2}\, dx - 2\int_0^3 \sqrt{9 - x^2}\, dy = -9\left(1 + \frac{\pi}{2}\right)$$

Hence, $\oint_{OABO} \mathbf{F} \bullet dl = -9\left(1 + \frac{\pi}{2}\right)$

Curl of A Vector Field



$$(\nabla\times\mathbf{A})_x = \lim_{\Delta y\Delta z\to 0}\frac{1}{\Delta y\Delta z}\left(\oint_{\substack{sides\\1,2,3,4}}\mathbf{A}\cdot dl\right), \mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

Side 1 : $dl = \mathbf{a}_z \Delta z$,

$$\mathbf{A}\cdot dl = A_z\left(x_0, y_0+\frac{\Delta y}{2}, z_0\right)\Delta z \approx \left(A_z(x_0,y_0,z_0)+\frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T}\right)\Delta z$$

Side 3 : $dl = -\mathbf{a}_z \Delta z$,

$$\mathbf{A}\cdot dl = -A_z\left(x_0, y_0-\frac{\Delta y}{2}, z_0\right)\Delta z \approx -\left(A_z(x_0,y_0,z_0)-\frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T}\right)\Delta z$$

$$\int_{\substack{sides\\1\text{ and }3}}\mathbf{A}\cdot dl = \left(\frac{\partial A_z}{\partial y}+\text{H.O.T.}\right)_{(x_0,y_0,z_0)}\Delta y\Delta z, \int_{\substack{sides\\2\text{ and }4}}\mathbf{A}\cdot dl = \left(-\frac{\partial A_y}{\partial z}+\text{H.O.T.}\right)_{(x_0,y_0,z_0)}\Delta y\Delta z$$

$$(\nabla\times\mathbf{A})_x = \frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}$$

Curl of A Vector Field

- $$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

- $$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- In general orthogonal curvilinear coordinates

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{a}_{u_1} & h_2 \mathbf{a}_{u_2} & h_3 \mathbf{a}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Curl of A Vector Field

- Example 2-21
  - Show that $\nabla \times \mathbf{A} = 0$ if

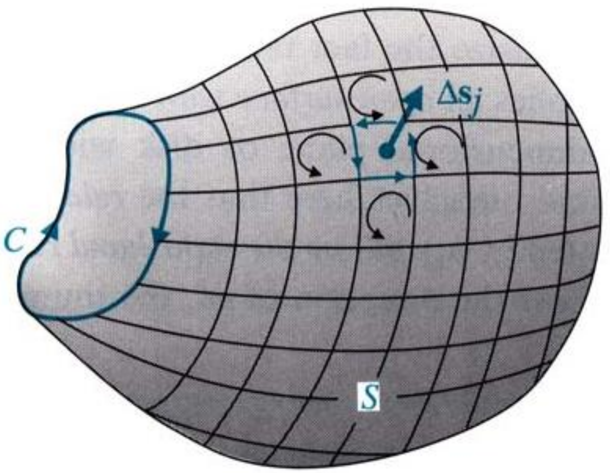
(a) $\mathbf{A} = \mathbf{a}_\phi (k/r)$ in cylindrical coordinates, where k is a constant, or

(b) $\mathbf{A} = \mathbf{a}_R f(R)$ in spherical coordinates, where $f(R)$ is any function of the radial distance R.

→ **A** *is* an *irrotational* or a *conservative* field

Stokes's Theorem

- $$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot dl$$



- $$(\nabla \times \mathbf{A})_j \cdot d\mathbf{s}_j = \oint_{C_j} \mathbf{A} \cdot dl$$

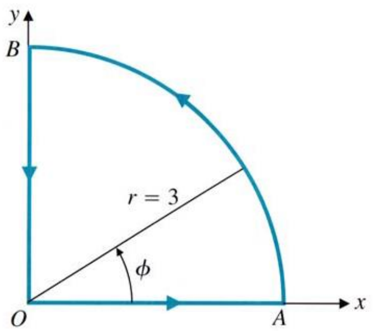
$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot \Delta \mathbf{s}_j = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \mathbf{A} \cdot dl \right) = \oint_C \mathbf{A} \cdot dl$$

- Requirement : **A** and its first derivatives exists and be continuous both on *S* and along *C*

Stokes's Theorem

- Example 2-22
  - Given $\mathbf{F} = \mathbf{a}_x xy - \mathbf{a}_y 2x$, verify Stokes's theorem.



$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -\mathbf{a}_z (2+x)$$

For given geometry and the designated direction of dl, $d\mathbf{s} = \mathbf{a}_z dxdy$

$$\int_S (\nabla \times \mathbf{F}) \bullet d\mathbf{s} = \int_0^3 \int_0^{\sqrt{9-x^2}} (\nabla \times \mathbf{F}) \bullet \mathbf{a}_z dxdy = \int_0^3 \left[\int_0^{\sqrt{9-x^2}} -(2+x)\, dx \right] dy$$

$$= -\int_0^3 \left[2\sqrt{9-x^2} + \frac{9-y^2}{2} \right] dy$$

$$= -\left[y\sqrt{9-y^2} + 9\sin^{-1}\frac{y}{3} + \frac{9}{2}y - \frac{y^3}{6} \right]_0^3 = -9\left(1 + \frac{\pi}{2}\right)$$

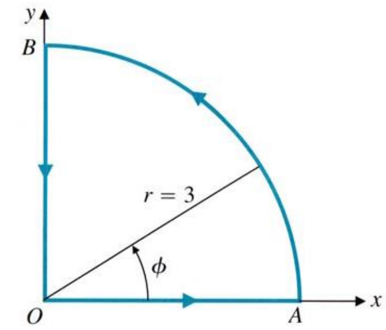
Stokes's Theorem

- Exercise

$$\mathbf{F} = \mathbf{a}_r \sin\phi + \mathbf{a}_\phi 3\cos\phi$$

$a)$ determine $\oint_{OABO} \mathbf{F} \bullet dl$

$b)$ find $\nabla \times \mathbf{F}$ and verify Stokes's theorem



  - a) 6
  - b) $\mathbf{a}_z \dfrac{2}{r} \cos\phi$

Null Identities

- The curl of the gradient of any scalar field is identically zero.

- $\nabla \times (\nabla V) = 0$

$$\int_S \nabla \times (\nabla V) \cdot d\mathbf{s} = \oint_C \nabla V \cdot dl = 0 \leftarrow dV = \nabla V \cdot dl$$

- If a vector is curl-free, then it can be expressed as the gradient of a scalar field.

$$\nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla V$$

Null Identities

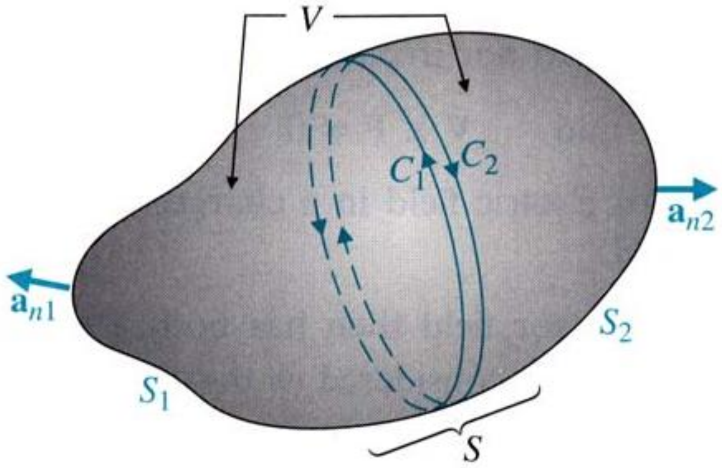
- The divergence of the curl of any vector field is identically zero.

- $$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\int_V \nabla \cdot (\nabla \times \mathbf{A}) d\upsilon = \oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

$$= \oint_{S_1} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n1} ds + \oint_{S_2} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n2} ds$$

$$= \oint_{C_1} \mathbf{A} \cdot d\ell + \oint_{C_2} \mathbf{A} \cdot d\ell$$



$$\nabla \cdot \mathbf{B} = 0 \rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

Laplace equation

- Laplacian = "the divergence of the gradient of " $\nabla^2 = \nabla \bullet \nabla$

- $$\nabla^2 V = \nabla \bullet \nabla V = \left(\mathbf{a_x} \frac{\partial}{\partial x} + \mathbf{a_y} \frac{\partial}{\partial y} + \mathbf{a_z} \frac{\partial}{\partial z} \right) \bullet \left(\mathbf{a_x} \frac{\partial V}{\partial x} + \mathbf{a_y} \frac{\partial V}{\partial y} + \mathbf{a_z} \frac{\partial V}{\partial z} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplace equation

- Laplacian in orthogonal curvlinear coordinates (u_1,u_2,u_3)

$$\nabla^2 V = \nabla \bullet (\nabla V) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right]$$

$$\leftarrow \nabla \bullet \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

- Laplace eqn & Poisson eqn.

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}, \ \nabla^2 V = 0$$

Field Classification and Helmholtz's Theorem

- Solenoidal and irrotational if

  $\nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} = 0$ (static electric field in a charge-free region)

- Solenoidal but not irrotational if

  $\nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} \neq 0$ (A steady magnetic field in a current-carrying conductor)

- Irrotational but not solenoidal if

  $\nabla \times \mathbf{F} = 0, \; \nabla \cdot \mathbf{F} \neq 0$ ⇨ A static electric field in a charged region

- Neither solenoidal nor irrotational if

  $\nabla \cdot \mathbf{F} \neq 0, \; \nabla \times \mathbf{F} \neq 0$ ⇨ An electric field in a charged medium with a time-varying magnetic field

- A vector field is determined *if both its divergence and its curl are specified everywhere.*

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

Some useful vector formulas

- Need to be memorized

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla(\psi V) = \psi\nabla V + V\nabla\psi$$

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \nabla\psi\cdot\mathbf{A}$$

$$\nabla\times(\psi\mathbf{A}) = \psi\nabla\times\mathbf{A} + \nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\times\nabla V = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

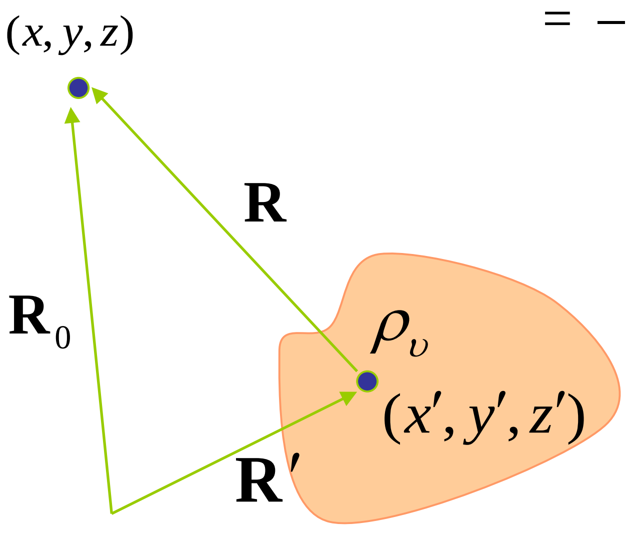
Some useful vector formulas

- Position vector, **R** $\quad R = |\mathbf{R}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$
- Gradient of 1/*R*

$$\frac{1}{R} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}$$

$$\nabla\left(\frac{1}{R}\right) = \mathbf{a}_x \frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \mathbf{a}_y \frac{\partial}{\partial y}\left(\frac{1}{R}\right) + \mathbf{a}_z \frac{\partial}{\partial z}\left(\frac{1}{R}\right)$$

$$= -\frac{\mathbf{a}_x(x-x') + \mathbf{a}_y(y-y') + \mathbf{a}_z(z-z')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} = -\frac{\mathbf{R}}{R^3} = -\frac{\mathbf{a}_R}{R^2}$$



$$\nabla\left(\frac{1}{R}\right) = -\frac{\mathbf{a}_R}{R^2}$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}$$

H.W.

- 2장
  - 1(e,f,g,h), 2, 16,17,18,19,20,24,26,27,28,30,32,34,36