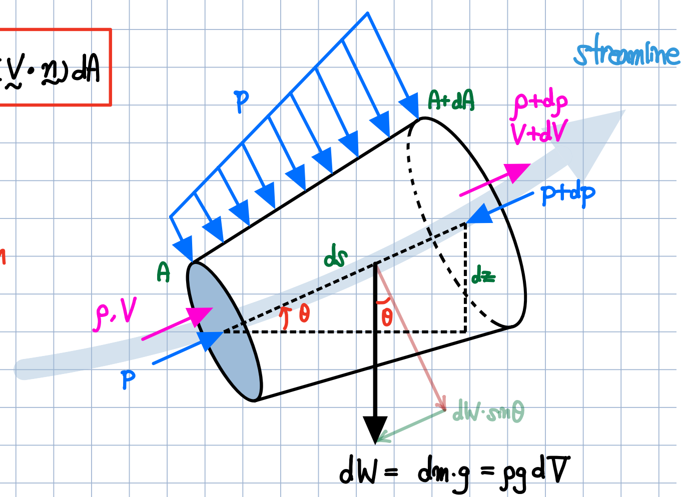
24강. Frictionless Flow : The Bernoulli Equation (along a streamline in frictionless flow)

Daniel Bernoulli (1738) ⟶ Leonhard Euler (1755)

For this elemental control volume,

Reynolds Transport Theorem : $\boxed{\dfrac{dB_{sys}}{dt} = \dfrac{d}{dt}\left(\int_{cv}\beta\rho d\mathcal{V}\right) + \int_{cs}\beta\cdot\rho\cdot(\underset{\sim}{V}\cdot\underset{\sim}{n})dA}$



1. Conservation of Mass ⟶ $B = m$, $\beta = B/m = 1$

$$\frac{d}{dt}(m)_{sys} = 0 = \frac{d}{dt}\left(\int_{cv}\rho d\mathcal{V}\right) + \int_{cs}\rho\cdot(\underset{\sim}{V}\cdot\underset{\sim}{n})dA \longleftarrow \rho AV = \dot{m}$$

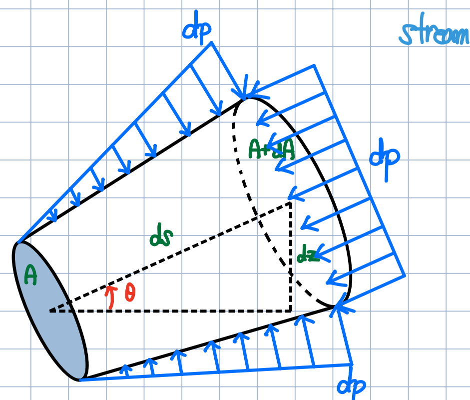
$$\frac{d}{dt}\left(\int_{cv}\rho d\mathcal{V}\right) + \dot{m}_{out} - \dot{m}_{in} = 0$$

$$\frac{\partial\rho}{\partial t}d\mathcal{V} + d\dot{m} = 0$$

$$d\dot{m} = -\frac{\partial\rho}{\partial t}\boxed{d\mathcal{V}} = -\frac{\partial\rho}{\partial t}\cdot Ads$$

$$\approx Ads + \cancel{dAds} = Ads$$

2. The Linear Momentum Relation ⟶ $B = m\underset{\sim}{V}$, $\beta = \underset{\sim}{V}$



$$\frac{d}{dt}(m\underset{\sim}{V})_{sys} = \frac{d}{dt}\left(\int_{cv}\underset{\sim}{V}\rho d\mathcal{V}\right) + \int_{cs}\underset{\sim}{V}\rho(\underset{\sim}{V}\cdot\underset{\sim}{n})dA$$

$$\Sigma\boxed{dF_s} = \frac{d}{dt}\left(\int_{cv}V\rho d\mathcal{V}\right) + (\dot{m}V)_{out} - (\dot{m}V)_{in}$$

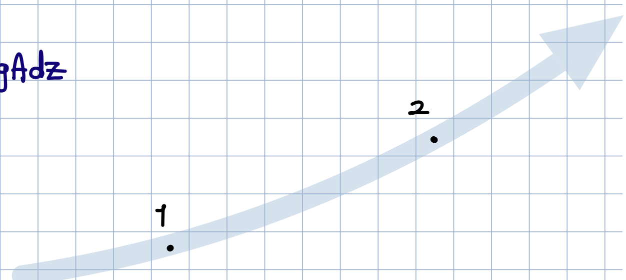
s : streamline

$$\Sigma\boxed{dF_s} = \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

1. gravity : $dF_{s,grav} = -dW\cdot\sin\theta = -\rho gAds\cdot\sin\theta = -\rho gAdz$

2. pressure : $\frac{1}{2}\cdot dp\cdot dA - (A+dA)\cdot dp \approx -A\cdot dp$ ★★

3. ~~Viscosity~~ : neglected due to frictionless assumption



$$\therefore \Sigma dF_s = -\rho gAdz - Adp = \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

$$= \frac{\partial\rho}{\partial t}\cdot VAds + \rho\cdot\frac{\partial V}{\partial t}\cdot Ads + d\dot{m}V + \dot{m}dV \longleftarrow d\dot{m} = -\frac{\partial\rho}{\partial t}Ads$$

$$= \cancel{\frac{\partial\rho}{\partial t}\cdot VAds} + \rho\cdot\frac{\partial V}{\partial t}\cdot Ads - \cancel{\frac{\partial\rho}{\partial t}AdsV} + \dot{m}dV \longleftarrow \dot{m} = \rho AV$$

$$-\rho gAdz - Adp = \rho A\cdot\frac{\partial V}{\partial t}ds + \rho AVdV \longleftarrow \rho A\text{로 나누면}$$

$$-gdz - \frac{dp}{\rho} = \frac{\partial V}{\partial t}ds + V\cdot dV \longrightarrow \boxed{\frac{\partial V}{\partial t}ds + \frac{dp}{\rho} + VdV + gdz = 0}$$

Ans.

i) Unsteady frictionless flow along a streamline : $\boxed{\int_1^2\frac{\partial V}{\partial t}ds + \int_1^2\frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0}$

ii) Steady Incompressible flow : $\dfrac{p_2 - p_1}{\rho} + \dfrac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$

$\dfrac{\partial V}{\partial t} = 0$ ρ = constant

$$\boxed{\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 = \text{const}}$$

Ans.